EXECUTIVE SUMMARY

This document is composed of two main parts. In the first one (chapter II), all the equations solved in TELEMAC-2D are derived: different formulations of shallow water (or Saint-Venant) equations, tracer equation, turbulence models, etc. Some basics on hydraulics are also presented: hydraulic jumps, characteristic curves, Froude number. The source terms added to the equations to deal with friction, wind, Coriolis force, and atmospheric pressure are also detailed.

The latter part describes how to solve the equations. The finite element method, the discretisation in space and time are briefly recalled. The variational formulation leading to the final linear system is fully given. The numerical schemes used in TELEMAC-2D for solving the advection equations are also explained.

The implementation of the techniques described is not given in this document.

SYNTHESE

Cette note se compose de deux grandes parties. La premi• re (chapitre II) Žtablit toutes les Žquations rŽsolues dans le code TELEMAC-2D : Žquations de Saint-Venant dans leurs diverses formulations, Žquation d'un traceur passif, mod• les de turbulence, etc. Des notions d'hydraulique sont Žgalement fournies : ressauts, courbes caractŽristiques, nombre de Froude. Les divers termes sources ajoutŽs aux Žquations de Saint-Venant et prenant en compte les frottements, la contrainte due au vent, la force de Coriolis et la pression atmosphŽrique sont Žgalement dŽtaillŽs.

La deuxi• me partie (chapitre III) dŽcrit la rŽsolution des Žquations : quelques notions sur les ŽlŽments finis, la discrŽtisation en temps et en espace sont rappelŽes. Tous les dŽtails de la formulation variationnelle menant au syst• me linŽaire final sont donnŽs. Les diffŽrents schŽmas numŽriques utilisŽs dans TELEMAC-2D pour la rŽsolution des Žquations de transport sont Žgalement expliquŽs.

Par contre la mise en oeuvre informatique de ces techniques ne rel• ve pas de ce document.

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1. INTRODUCTION:

The TELEMAC system is a set of finite element programs designed for the open channel flows which uses a string of common processes (digitization and graphics); it contains two and three dimension modules for the study of currents, sedimentation, waves and water quality.

Forming the core of this system, TELEMAC-2D, the first module of this package, is a program for the solution of the two dimensional Saint-Venant equations. The water depth and the velocity averaged on the vertical are the main variables, but the transport of a passive tracer as well as turbulence can be taken into consideration.

It uses triangular finite element discretization, it can also work with quadrilateral elements. TELEMAC-2D can be used for numerous studies in fluvial and maritime hydraulics, the test cases considered for the validation of the program give an idea of its fields of application.

The present note describes the basic equations and the solution methods used, but does not give the details of computer techniques. Another part of the literature on the program is given with the validation document [1] and the user's guide [2].

The first part is devoted to the equations with details of the elements that go into creating a model: the conservative form, the existence of characteristic curves and hydraulic jumps, as well as the hypotheses and approximations. To neglect or be unaware of the basis and principles of Saint-Venant equations can lead to an improper statement of the problem or to unrealistic boundary conditions, which would give wrong results. In the preliminary stages of the study the validity of hypotheses and the ability of the code should be seriously evaluated with regard to the stated problem.

This first part can be referred to to understand better the significance of the physical parameters offered as options in TELEMAC-2D.

The second part expounds the algorithms for the solutions of equations set out earlier. The general lines of discretization and variational formulations are presented. The difficulties, the theoritical limitations and the unresolved problems are also cited. This document forms the basis for future improvements, and all contributions to this end will be welcome.

TELEMAC is built up according to the wishes of the users by successive addition of new functions. As this could lead to an increasing complexity, a constant effort to simplify the program was necessary. The release 3.0 includes changes made to limit its size and to preserve the possibilities of further development in future:

Withdrawal of the option celerity-velocity.Giving up of specific treatment for the quadrilateral method.New methods of working with Mercator projections.

The important new features include a new option for treating tidal flats (masking certain elements) new advection schemes, and consideration of horizontal density gradient.

II. THE EQUATIONS:

II.1 INTRODUCTION

BarrŽ de Saint-Venant equations (1871) are of the greatest importance in maritime and fluvial hydraulics and govern shallow water open channel flows. These are derived from Navier-Stokes equations by assuming certain simplifying hypotheses and are used in such wide areas as environmental studies, the calculation of tides and storm surges, impact and stability of structures, sedimentology and the study of floods.

One of the restrictive hypotheses of these equations is that the wavelength should be large in relation to the depth, therefore the computation of waves is not in their domain of validity.

Saint-Venant equations and their different formulations are presented here to bring out the phenomena that are modelled and the precautions implied in the numerical analysis.

II.2 SAINT-VENANT EQUATIONS

Saint-Venant equations are derived from Navier-Stokes equations by taking the vertical average in which, however, the non linear terms necessitate certain assumptions and approximations. These assumptions and approximations form the limits and are responsible for the restricted domain of validity of the equations.

II.2.1 Notations and basics of geometry

An orthogonal Cartesien reference frame (x,y,z) is selected where the x and y axes form the horizontal plane and gravity acts in the negative z direction.



Zf and Z are the elevations of the bottom and the free surface respectively.

The depth of the water h is equal to Z-Zf.

The elevation of the bottom is given, the depth of the water is generally an unknown.

The time is denoted by t.

Normal to the bottom and to the free surface:

All points on the free surface are uniquely defined as a function of x, y and t, which precludes breaking waves. The surface is defined by an equation of the form: z=Z(x,y,t) which may also be written as:

 $\Phi(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}) = 0 \text{ with } \Phi(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}) = \mathbf{Z}(\mathbf{x},\mathbf{y},\mathbf{t}) - \mathbf{z}.$

The normal to the free surface in the positive z direction (outward pointing normal for the volume of water) can be expressed as the vector $\vec{n} = \text{grad}(\Phi)$. The components of this vector are:

$$\vec{n} \left(\begin{array}{c} \frac{\partial Z}{\partial x} \\ \frac{\partial Z}{\partial y} \\ -1 \end{array} \right)$$

Note that the norm of \vec{n} is not equal to 1.

The normal to the bottom can be expressed similarly by substituting Zf for Z and with a minus sign to get an outward normal. The bottom is also a function of x and y. A vertical bottom can only be represented by a steep slope.

 Φ which describes the movement of any point on the free surface is always 0 whereby the derivative $\frac{d\Phi}{dt}$ is 0 (this is the Lagrangian approach, which in physical terms means that over a period of time a particle of water on the free surface remains attached to the surface). By denoting the velocity of a point on the surface by \vec{U}^s (composed of U_1^s , U_2^s et U_3^s), the equation $\frac{d\Phi}{dt} = 0$ can be rewritten using Euler's variables as: $\frac{\partial Z}{\partial t} + U_1^s \frac{\partial Z}{\partial x} + U_2^s \frac{\partial Z}{\partial y} - U_3^s = 0$, which gives: $\frac{\partial Z}{\partial t} + \vec{U}^s \cdot \vec{n} = 0$

At the bottom we obtain similarly: $\frac{\partial Zf}{\partial t} + \vec{U}^{f}$. $\vec{n} = 0$. The hypotheses of the bottom constant

over time is not required.

This relation will be used in establishing the equations that describe the impermeability of the bottom and the free surface.

II.2.2 Navier-Stokes equations for averaged Reynolds number

The fluid is assumed to be Newtonian.

The starting equations are the Navier-Stokes equations for variable density. The conservative form of the equations is chosen which would result in the conservative form of Saint-Venant equations:

Continuity:

 $\frac{\partial(\rho U_i)}{\partial x_i} = 0$

Momentum :

 $\frac{\partial(\rho U_i)}{\partial t} + \frac{\partial(\rho U_i U_j)}{\partial x_j} = - \ \frac{\partial p}{\partial x_i} + \ \frac{\partial}{\partial x_j} \left(\tau_{ij} + R_{ij}\right) + \rho F_i + \rho g_i$

The indices i and j vary between 1 and 3. Einstein tensor notation is used hereunder. Repetition of the indices imply a summation, for example:

$$\frac{\partial(\rho U_i U_j)}{\partial x_j} \quad \text{represents} \quad \frac{\partial(\rho U_i U_1)}{\partial x_1} + \frac{\partial(\rho U_i U_2)}{\partial x_2} + \frac{\partial(\rho U_i U_3)}{\partial x_3}$$

Fi represents forces applied in the volume of water, other than pressure and weight (see chapter V)

gi represents gravity($g_1 = g_2 = 0, g_3 = -g$)

p is the pressure (strictly speaking the spherical part of the constraint tensor).

$$\begin{split} \tau_{ij} &= \mu(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}) \quad \text{corresponds to the viscous part of the constraint tensor. The molecular} \\ \text{viscosity } \nu, \text{ in } m^2/\text{s}, \text{ is equal to } \frac{\mu}{\rho}. \text{ It should be noted that, if } \rho \text{ is constant, the trace of the tensor} \\ \text{is 0 due to the continuity equation.} \end{split}$$

The stresses
$$\tau_{ij}$$
 appear in the expression as $\frac{\partial}{\partial x_j} (\tau_{ij})$, that is to say $\frac{\partial}{\partial x_j} (\mu(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}))$. Using the continuity equation, and if ρ is constant an equivalent form is $\frac{\partial}{\partial x_j} (\mu \frac{\partial U_i}{\partial x_j})$.

 $R_{ij} = -\rho U_i U_j$ is the Reynolds tensor due to turbulence. Strictly, the above equations are Reynolds equations where velocity and pressure are stochastic means devoid of turbulent fluctuations. The bar indicates the stochastic mean (which supposes a hypothesis of ergodicity) and the U_i are the fluctuations of the velocity. If the fluid is incompressible, the Reynolds average may be used else, for compressible fluids, other averaging procedures (such as Favre) must be used. ρ is below the bar and may be moved out only if the density does not undergo turbulent fluctuations. These additional terms which do not figure in Navier-Stokes equations proper arise from non linear terms where turbulent fluctuations cannot cancelled out. In the momentum equation following the stochastic process, we obtain (the x componant for example):

$$- \overline{\rho U_1^{'} \frac{\partial U_1^{'}}{\partial x}} - \overline{\rho U_2^{'} \frac{\partial U_1^{'}}{\partial y}} - \overline{\rho U_3^{'} \frac{\partial U_1^{'}}{\partial z}}$$

which is expressed as:

div (-
$$\overline{\rho U_1' \vec{U}}$$
)

In most of the turbulence models, the terms:

 $\frac{1}{\rho} \frac{\partial}{\partial x_j} (\tau_{ij} + R_{ij}) \text{ are expressed as: } \frac{\partial}{Z x_j} ((\nu + \nu_t) [\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}]) \text{ where } \nu_t \text{ is the turbulent viscosity.}$ $\nu_t \text{ varies with time and space and is obtained from the turbulence model. This formulation forms the hypothesis of Boussinesq on the turbulent viscosity. Henceforth we will write <math>\nu_e = \nu_+ \nu_t$ (effective viscosity) and the diffusion terms as $\frac{\partial}{Z x_j} (\nu_e [\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}])$

We then suppose that the density is constant (variable density is studied in paragraph II.2.10), which finally leads to:

Continuity :

$$\frac{\partial U_i}{\partial x_i} = 0$$

Momentum

$$\frac{\partial U_i}{\partial t} + \frac{\partial U_i U_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial}{\partial x_j} (\tau_{ij} + R_{ij}) + F_i + g_i$$

II.2.3 Hypotheses, approximations, and rules of calculations

Hydrostatic pressure:

We will assume henceforth that in the following momentum equation the vertical acceleration caused by the pressure balances gravity. In this case, we say that the pressure is hydrostatic. This is linked to conditions of small vertical movements.

In fact, if this hypothesis is not absolutely necessary to develop Saint-Venant equations, it most certainly is to convert the pressure in terms of the water head.

Since the pressure, denoted by p(x, y, z), at a point of coordinates (x, y, z), is caused only by the water head above that point:

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} - g = 0$$
 where p (x, y, z) = - ρ g z + cste

The constant is chosen such that p(x, y, Z) = 0, where Z is the level of the free surface. We therefore assume that the atmospheric pressure is 0 (or is a constant almost equal to 0). (If it is not, we will take it into account later in the paragraph II.2.8.4). It follows then:

$$p(x, y, z) = \rho g(Z - z)$$

at the bottom: $p = \rho g (Z - Zf) = \rho g h$, h being the water depth.

Negligible vertical velocities:

The vertical velocity will be neglected in the Saint-Venant equations and will not have an equation. This approximation is linked to the hypothesis of hydrostaticity that requires the vertical accelerations to be insignificant.

Impermeability of the surface and of the bottom:

We further assume that there will be no transfer of water either through the bottom or from the surface, and that a particle of water located on one of these two interfaces will remain there. Following on from the equations established in paragraph II.2.1, we can write:

U₁ (x, y, Z)
$$\frac{\partial Z}{\partial x}$$
 + U₂ (x, y, Z) $\frac{\partial Z}{\partial y}$ + $\frac{\partial Z}{\partial t}$ = U₃ (x, y, Z)

and

U₁ (x, y, Zf)
$$\frac{\partial Zf}{\partial x}$$
 + U₂ (x, y, Zf) $\frac{\partial Zf}{\partial y}$ + $\frac{\partial Zf}{\partial t}$ = U₃ (x, y, Zf)

Note:

We will explore later the source terms due to the influx of water either from the surface or from the bottom. In this case, impermeability will no longer exist.

Rule of Leibnitz:

The derivative of an integral with variable limits gives rise to a derivative inside the integral and to the flux terms according to the formula:

$$\frac{\partial}{\partial x} \int_{z_{f}}^{z} F dz = \int_{z_{f}}^{z} \frac{\partial F}{\partial x} dz + F(x,y,Z) \frac{\partial Z}{\partial x} - F(x,y,Zf) \frac{\partial Zf}{\partial x}$$

If we substitute $f = \frac{1}{h} \int_{z_f}^{z} F dz$, (f represents the average F over the vertical), we will get by applying the preceding formula to derivatives with respect to x, y, and z:

$$\overrightarrow{\text{grad}}(\text{hf}) = \int_{Z_f}^{Z} \overrightarrow{\text{grad}}(F) \, dz + F(x,y,Z) \vec{n}_{\text{surface}} - F(x,y,Zf) \vec{n}_{\text{fond}}$$

where $\vec{n}_{surface}$ and \vec{n}_{fond} are, let us recall, non-unitary vectors.

In a similar way, if F is a vector:

~

$$\operatorname{div}(\mathbf{h}\,\vec{f}) = \int_{Z_f}^{Z} \operatorname{div}(\vec{F}) \, \mathrm{d}z + \vec{F}(x,y,Z) \, . \quad \vec{n}_{\text{surface}} - \vec{F}(x,y,Zf) \, . \quad \vec{n}_{\text{fond}}$$

For a derivative in time, and keeping in mind that:

$$\frac{\partial Z}{\partial t} + \vec{U} \cdot \vec{n} = 0$$
 and $\frac{\partial Zf}{\partial t} + \vec{U} \cdot \vec{n} = 0$, we get:

$$\frac{\partial(hf)}{\partial t} = \int_{Zf}^{Z} \frac{\partial F}{\partial t} dz - F(x,y,Z) \vec{U} \cdot \vec{n}_{surface} + F(x,y,Zf) \vec{U} \cdot \vec{n}_{fond}$$

II.2.4 The average of Navier-Stokes equations

II.2.4.1 Principle:

The Navier-Stokes equations (or Reynolds equations if we consider the turbulence) at constant density and with hydrostatic pressure will be averaged over the vertical by integrating from the bottom to the surface. Apart from the already defined water depth, two new variables appear:

$$u = \frac{1}{h} \int_{z_f}^z U_1 dz$$
 and $v = \frac{1}{h} \int_{z_f}^z U_2 dz$

These averages over the vertical of the horizontal components of the velocity vector will themselves be called velocity, or «components of the velocity vector».

II.2.4.2 Average of the continuity equation:

The continuity equation becomes:

$$\int_{z}^{z} \left(\frac{\partial U_{1}}{\partial x} + \frac{\partial U_{2}}{\partial y} + \frac{\partial U_{3}}{\partial z} \right) dz = 0$$

By using Leibnitz's rule, it follows:

$$\int_{z_{f}}^{z} \left(\frac{\partial U_{1}}{\partial x} + \frac{\partial U_{2}}{\partial y} + \frac{\check{Z}U_{3}}{\partial z}\right) dz = \frac{\partial}{\partial x} \int_{z_{f}}^{z} U_{1} dz - U_{1} (x, y, Z) \frac{\partial Z}{\partial x} + U_{1} (x, y, Zf) \frac{\partial Zf}{\partial x}$$
$$+ \frac{\partial}{\check{Z}y} \int_{z_{f}}^{z} U_{2} dz - U_{2} (x, y, Z) \frac{\partial Z}{\partial y} + U_{2} (x, y, Zf) \frac{\partial Zf}{\partial y}$$
$$+ U_{3} (Z) - U_{3} (Zf) = 0$$

And, considering the conditions of impermeability:

$$\frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) + \frac{\partial Z}{\partial t} - \frac{\partial Zf}{\partial t} = 0$$

or finally:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (hu) + \frac{\partial}{\partial y} (hv) = 0$$
 or $\frac{\partial h}{\partial t} + div (h\vec{u}) = 0$

We remark straight away the analogy with the continuity equation of compressible fluids:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \left(\rho \vec{u} \right) = 0$$

in the Saint-Venant equations the depth plays the role of the density .

II.2.4.3 Average of the momentum equation

We shall apply from the very beginning the hypothesis of hydrostatic pressure. The equation of U_1 , once developed, becomes:

$$\frac{\partial U_{1}}{\partial t} + \frac{\partial U_{1}^{2}}{\partial x} + \frac{\partial U_{1}U_{2}}{\partial y} + \frac{\partial U_{1}U_{3}}{\partial z} = -\frac{1}{\rho} \frac{\partial}{\partial x} \left[\rho g \left(Z - z\right)\right] \\ + \frac{1}{\rho} \left[\frac{\partial}{\partial x} \left(\tau_{11} + R_{11}\right) + \frac{\partial}{\partial y} \left(\tau_{12} + R_{12}\right) + \frac{\partial}{\partial z} \left(\tau_{13} + R_{13}\right)\right] + F_{x}$$

Let us detail the results from integrating each of these terms:

Time derivative:

$$\int_{Z_{f}}^{Z} \frac{\partial U_{1}}{\partial t} dz = \frac{\partial (hu)}{\partial t} - U_{1} (Z) \frac{\partial Z}{\partial t} + U_{1} (Zf) \frac{\partial Zf}{\partial t}$$

advection terms:

$$\int_{z_{f}}^{z} \frac{\partial U_{1}^{2}}{\partial x} dz = \frac{\partial}{\partial x} \int_{z_{f}}^{z} U_{1}^{2} dz - U_{1}^{2} (Z) \frac{\partial Z}{\partial x} + U_{1}^{2} (Zf) \frac{\partial Zf}{\partial x}$$
$$\int_{z_{f}}^{z} \frac{\partial U_{1}U_{2}}{\partial y} dz = \frac{\partial}{\partial y} \int_{z_{f}}^{z} U_{1} U_{2} dz - U_{1} (Z) U_{2} (Z) \frac{\partial Z}{\partial y} + U_{1} (Zf) U_{2} (Zf) \frac{\partial Zf}{\partial y}$$

The last two terms produce non-linearities. Let us develop for example the expression in U_1U_2 :

$$\frac{\partial}{\partial y} \int_{z_{f}}^{z} U_{1} U_{2} dz = \frac{\partial}{\partial y} \int_{z_{f}}^{z} (u + U_{1} - u) (v + U_{2} - v) dz$$

$$= \frac{\partial}{\partial y} \int_{z_{f}}^{z} u v dz + \frac{\partial}{\partial y} \int_{z_{f}}^{z} u (U_{2} - v) + v (U_{1} - u) dz + \frac{\partial}{\partial y} \int_{z_{f}}^{z} (U_{1} - u) (U_{2} - v) dz$$
$$= \frac{\partial}{\partial y} (h u v) + \frac{\partial}{\partial y} \int_{z_{f}}^{z} (U_{1} - u) (U_{2} - v) dz$$

The term $\frac{\partial}{\partial y} \int_{z_f} (U_1 - u) (U_2 - v) dz$ is not nil when there are heterogeneities in the velocity

over the vertical. These terms, called those of dispersion, are generally interpreted as an additional diffusion. In fact, they take into account the fluctuations around the mean (here, spatial) and are analogous to Reynolds stress. In reality, these are advection terms arising from the fact that the movement of the water is due to a real current and not due to a current averaged over the vertical. Depending on the depth, all particles of water are not carried at the same velocity whereas in the Saint-Venant equations we choose an average current. We shall select here an approach with diffusion, with a tensor that shall add naturally to the Reynolds tensor. Henceforth, v_e shall include the molecular viscosity, the turbulence, and the dispersion.

$$\int_{Zf}^{z} \frac{\partial U_{1}U_{3}}{\partial z} dz = U_{1} (Z) U_{3} (Z) - U_{1} (Zf) U_{3} (Zf)$$

Pressure gradient:

 $\int_{z_{\rm f}}^{z} -\frac{1}{\rho} \frac{\partial}{\partial x} \left[\rho g \left(Z - z \right) \right] dz = -h g \frac{\partial Z}{\partial x}$ This is true only if the density does not change over the

horizontal. These variations will be taken into account in paragraph II.2.10.

Diffusion terms:

 $+\frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(\tau_{11}+R_{11}\right)+\frac{\partial}{\partial y}\left(\tau_{12}+R_{12}\right)+\frac{\partial}{\partial z}\left(\tau_{13}+R_{13}\right)\right]: \text{ let us examine, for example, viscous friction, written in the form }\frac{1}{\rho}\operatorname{div}(\vec{\tau}), \text{ where }\vec{\tau} \text{ is a vector with components }(\tau_{11}, \tau_{12}, \tau_{13}).$

If we assume that the density does not vary over the vertical, it can be removed from within the integral and the formula of Leibnitz gives:

$$\frac{1}{\rho} \int_{Z_f}^{z} \operatorname{div}(\vec{\tau}) \, dz = \frac{1}{\rho} \operatorname{div}(\int_{Z_f}^{z} \vec{\tau} \, dz) - \frac{1}{\rho} \vec{\tau}_{\text{surface}} \cdot \vec{n}_{\text{surface}} + \frac{1}{\rho} \vec{\tau}_{\text{fond}} \cdot \vec{n}_{\text{fond}}$$

The last two terms of the last equation represent surface and bottom constraints caused, for example, by the wind or the friction at the bottom. These terms will be studied later along with source terms (chapter II.2.8).

Remark on the way the term $\operatorname{div}(\vec{\tau})$ is written:

According to Boussinesq's model,
$$\vec{\tau}$$
 is a vector with these components:

$$\begin{pmatrix}
\mu_{e} \left(\frac{\partial U_{1}}{\partial x} + \frac{\partial U_{1}}{\partial x}\right) \\
\mu_{e} \left(\frac{\partial U_{1}}{\partial y} + \frac{\partial U_{2}}{\partial x}\right) \\
\mu_{e} \left(\frac{\partial U_{1}}{\partial z} + \frac{\partial U_{3}}{\partial x}\right)
\end{pmatrix}$$

where $\mu_e = \rho v_e$. Even though this is valid only when μ_e is constant in space (which is not the case with the turbulence model k-epsilon, and even less so when μ_e includes dispersion), we write div(τ) as div($\mu_e \operatorname{grad}(U_1)$). This approximation will avoid a coupling of the components of the velocity in the final equations, and conserves, in spite of everything, a tensorial form of the viscosity terms. Tests with other programmes preceding TELEMAC (notably ULYSSE) have shown that this coupling of the velocity components could have been ignored. Knowing that the effective viscosity includes the dispersion terms, we can say that we have chosen a model rather than an approximation, the expression of τ given by Boussinesq itself having been obtained from an analogy with molecular viscosity.

Before arriving at a final form of our diffusion terms, we shall have to make one more

approximation. We have to consider that
$$\int_{z_f} \overrightarrow{\text{grad}}(U_1) \, dz \approx h \overrightarrow{\text{grad}}(u),$$

The term $\frac{1}{\rho} \operatorname{div}(\int_{z_f}^z \overrightarrow{\tau} \, dz)$ is finally written : $\frac{1}{\rho} \operatorname{div}(h\mu \overrightarrow{\text{grad}}(u)).$

Assuming, in addition, that the density remains constant, the diffusion term is finally written as: div(h $\nu_e \operatorname{grad}(u)$).

In this expression, v_e is traditionally called «turbulent diffusion». This is the key word «VELOCITY DIFFUSIVITY». We have to remember that in the Saint-Venant equations, this term includes the dispersion. v_e can then vary substantially, from less than 1 to more than 500 m²/s. This imposes a practical standardization on every application if the model of turbulence is not used.

 $\begin{array}{ccc} \underline{Source \ terms \ and \ volumic \ forces:} \\ F_{x} & : & h \ F_{x} \end{array} \quad (terms \ assumed \ constant \ over \ the \ vertical) \end{array}$

Until we study them more precisely in chapter II.2.8, we shall include in these volumic forces the friction forces and the terms:

 $-\frac{1}{\rho}\vec{\tau}_{surface} \cdot \vec{n}_{surface} + \frac{1}{\rho}\vec{\tau}_{fond} \cdot \vec{n}_{fond}$ which shall be now omitted.

Recapitulation:

The sum of these terms, many of which disappear in view of the impermeability of the bottom and of the free surface, gives the following equation:

$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x} (huu) + \frac{\partial}{\partial y} (huv) = -hg \frac{\partial Z}{\partial x} + hF_x + div(h\nu_e \overrightarrow{grad}(u))$$

The U₂ equation gives:

$$\frac{\partial (hv)}{\partial t} + \frac{\partial}{\partial x} (huv) + \frac{\partial}{\partial y} (hvv) = -hg \frac{\partial Z}{\partial y} + hF_y + div(hve \overrightarrow{grad}(v))$$

Since the vertical velocity is assumed as insignificant, the U3 equation is not retained (it was already used in the hypothesis of hydrostaticity).

Along with the continuity equations, these equations form part of the bidimensional equations of BarrŽ de Saint-Venant in a form called «conservative.»

II.2.5 Different forms of equations

II.2.5.1 Conservative form

The equations obtained in the previous chapter are rewritten by taking: $Q_x = hu$, $Q_y = hv$. \vec{Q} is the vector with Q_x and Q_y as components. The discharge Q is in fact expressed here in m²/s.

Continuity:

 $\frac{\partial h}{\partial t} + \operatorname{div}(\vec{Q}) = 0$

Momentum :

$$\frac{\partial Q_x}{\partial t} + \operatorname{div}(\vec{u} Q_x) = -hg \frac{\partial Z}{\partial x} + h F_x + \operatorname{div}(h v_e \overrightarrow{\operatorname{grad}} u)$$
$$\frac{\partial Q_y}{\partial t} + \operatorname{div}(\vec{u} Q_y) = -hg \frac{\partial Z}{\partial y} + h F_y + \operatorname{div}(h v_e \overrightarrow{\operatorname{grad}} v)$$

Source terms in the continuity equation:

In the case of an intake or outflow on the bottom, or if we consider sub-soil seepage, the right side of the continuity equation is no longer 0 but is equal to a term that we shall call «Sce» for «Source». Sce is expressed in m/s and is positive for an influx of water in the domain. The relation between Sce and an inflow will only be clear when we do a variational formulation. In fact, the discharge of a source Q_{sce} of a point i, considered positive for an influx into the domain, will equal:

 $Q_{sce} = Sce \int_{\Omega} \phi_i d\Omega$ that is to say, Sce multiplied by the «volume» of the test function at point i.

The continuity equation will then become:

$$\frac{\partial h}{\partial t} + \operatorname{div}(\vec{Q}) = \operatorname{Sce}$$

In the same manner, we have to consider in the momentum equations the contribution of the source, expressed by vector components Mt(x) and Mt(y) (where Mt stands for «Momentum») with dimensions m^2/s^2 . These terms are equal, respectively, to u_{sce} Sce and v_{sce} Sce, where u_{sce} and v_{sce} are the components of the velocity of the source. In the case of soil seepage, these terms are assumed nil.

II.2.5.2 Non-conservative form

In theory, only a rigorous application of conservative equations will lead to a good grasp of the problem considering the discontinuities that can arise in the solution (hydraulic jumps). Other reasons (stability of the model, use of the characteristics method) have led us to adopt a different formulation with variables h and u, called the «depth-velocity» formulation. This formulation is called «non-conservative». Let us hasten to add that this terminology does not preclude construction of numerical models that preserve the water mass.

Using the conservative form of equations as a starting point, the derivatives of the products of functions are developed in the equations of momentum and of continuity.

The continuity equation becomes:

 $\frac{\partial h}{\partial t} + \vec{u} \cdot \vec{\text{grad}}(h) + h \operatorname{div}(\vec{u}) = \text{Sce}$

and the equation of u:

$$h\frac{\partial u}{\partial t} + u\frac{\partial h}{\partial t} + u\frac{\partial hu}{\partial x} + hu\frac{\partial u}{\partial x} + u\frac{\partial hv}{\partial y} + hv\frac{\partial u}{\partial y} = -hg\frac{\partial Z}{\partial x} + hFx + div(hv_e \overrightarrow{grad} u) + u_{sce} Sce$$

And, using the continuity equation:

$$h\frac{\partial u}{\partial t} + hu\frac{\partial u}{\partial x} + hv\frac{\partial u}{\partial y} = -hg\frac{\partial Z}{\partial x} + hFx + div(hv_e \overrightarrow{grad} u) - uSce + u_{sce}Sce$$

We divide both sides of the equation by h. And, working in the same manner for v, we arrive at:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -g \frac{\partial Z}{\partial x} + F_x + \frac{1}{h} \operatorname{div}(h \ v_e \ \overline{\text{grad}} \ u) + \frac{Sce}{h}(u_{sce} - u)$$
$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -g \frac{\partial Z}{\partial y} + F_y + \frac{1}{h} \operatorname{div}(h \ v_e \ \overline{\text{grad}} \ v) + \frac{Sce}{h}(v_{sce} - v)$$

The conservative and non-conservative forms of equations are not equivalent if discontinuities are present.

Remark:

If there is a source at the bottom, the terms $\frac{(u_{sce} - u) Sce}{h}$ and $\frac{(v_{sce} - v) Sce}{h}$ have to be added to the right hand sides. These terms are nil if the velocity of the source is the same as that of the current. We will henceforth include them in the expressions F_x and F_y and will not mention them explicitly.

The two main phenomena modelled by Saint-Venant equations are advection (or the transport by the current) and the propagation of long waves. The propagation is easily defined for waves of small heights where linearised equations are valid; the advection terms are negligible and ignored, and a flat bottom is assumed. This leads to, in dimension 1:

$$\frac{\partial u}{\partial t} + g \frac{\partial h}{\partial x} = 0$$
$$\frac{\partial h}{\partial t} + h_0 \frac{\partial u}{\partial x} = 0$$

 h_0 is the average depth

The combination:

$$-h_0 \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} + g \frac{\partial h}{\partial x} \right) + \frac{\check{Z}}{\partial t} \left(\frac{\partial h}{\partial t} + h_0 \frac{\partial u}{\partial x} \right)$$

gives :

$$\frac{\partial^2 \mathbf{h}}{\partial t^2} - \mathbf{g} \mathbf{h}_0 \frac{\partial^2 \mathbf{h}}{\partial x^2} = 0$$

And, similarly:

$$\frac{\partial^2 u}{\partial t^2} - gh_0 \frac{\partial^2 u}{\partial x^2} = 0$$

These two equations show that u and h propagate at velocity c, where $c^2 = gh_0$. From Euler, we know indeed that the solution of an equation of the type $\frac{\partial^2 F}{\partial t^2} - c^2 \frac{\partial^2 F}{\partial x^2}$ is :

$$f(x, t) = A(x - ct) + B(x + ct),$$

the form of functions A and B being determined by initial and boundary conditions.

In the absence of a current, the celerity of waves from the Saint-Venant equations is therefore:

$$c = \sqrt{gh}$$

This shows the celerity of long waves of small amplitude in linear theory. For a depth of 2000 m, the celerity is therefore 500 km/h. This celerity has been actually observed in tidal waves and in tsunami.

II.2.5.4 Celerity-velocity formulation

This formulation was the one used in the previous versions of TELEMAC-2D. It is still presented here because it shows the analogy between the Saint-Venant and the Euler equations, and because it is the most natural formulation to present characteristic curves.

This formulation is obtained by replacing depth h by $\frac{c^2}{g}$ in the equations. We get: Continuity:

$$2 c \frac{\partial c}{\partial t} + 2c u \frac{\partial c}{\partial x} + 2c v \frac{\partial c}{\partial y} + c^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

And, after divison by 2c:

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} + \frac{c}{2} \operatorname{div}(\vec{u}) = 0$$

Momentum :

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + 2c \frac{\partial c}{\partial x} = F_x - g \frac{\partial Zf}{\partial x} + \frac{1}{h} \operatorname{div} (h v_e \overrightarrow{\operatorname{grad}} \overrightarrow{u})$$
$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + 2c \frac{\partial c}{\partial y} = F_y - g \frac{\partial Zf}{\partial y} + \frac{1}{h} \operatorname{div} (h v_e \overrightarrow{\operatorname{grad}} v)$$

This form of the equations is often simpler to develop the exact solutions and has advantages for numerical resolution.

II.2.5.5 Analogies with Euler's equations:

Euler's equations govern the flow of ideal compressible fluids at constant entropy. They are written as:

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} + \frac{\gamma - 1}{2} c \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] = 0$$

 $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{2}{\gamma - 1} c \frac{\partial c}{\partial x} = 0$ $\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{2}{\gamma - 1} c \frac{\partial c}{\partial y} = 0$

 γ is the ratio of heat masses at constant pressure and constant volume.

Here, c is the celerity of the waves in a fluid, equal to $\sqrt{\frac{\gamma p}{\rho}}$, p being the pressure and ρ the

density.

There is a perfect analogy between the equations of Euler and of Saint-Venant (for the latter, $\gamma = 2$; for air, $\gamma = 1.405$).

This analogy underscores the fact that the Saint-Venant equations are formally for compressible fluids, the water depth being the counterpart of the varying density in Euler's equation. Nevertheless, the analogy stops at the left hand side of the equations; the source terms of the Saint-Venant equations present specific difficulties that lead to a quick divergence of the two approaches.

II.2.6 Characteristic curves

The method of characteristics arises from the geometric theory of quasi-linear differential equations. It gives a physical interpretation to certain problems of fluid mechanics and sometimes leads to an analytical solution. It also indicates the necessary number and nature of boundary conditions and can detect ill-defined problems. The concept of characteristics and of Riemann invariants in the Saint-Venant equations is therefore fundamental.

We shall start at the celerity-velocity formulation, linearised, in dimension 1, without diffusion nor source terms, and in a flat-bottomed channel:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + c \frac{\partial (2c)}{\partial x} = 0$$
$$\frac{\partial (2c)}{\partial t} + u \frac{\partial (2c)}{Zx} + c \frac{\partial u}{\partial x} = 0$$

The sum of the two equations is: $\frac{\partial}{\partial t}(u+2c) + (u+c)\frac{\partial}{\partial x}(u+2c) = 0$

and the difference :
$$\frac{\partial}{\partial t} (u - 2c) + (u - c) \frac{\partial}{\partial x} (u - 2c) = 0$$

These equations are of the form:

 $\frac{\partial A}{\partial t} + \frac{dx}{dt}\frac{\partial A}{\partial x} = 0 \quad \text{or} \quad \frac{dA}{dt} = 0 \quad \text{on the curves of equation} \quad \frac{dx}{dt} = u + c \quad \text{ou} \quad u - c$

A equals u + 2c or u - 2c. We have thus:

$$\frac{d}{dt}(u+2c) = 0$$
 on the curves C+ of equation $\frac{dx}{dt} = u + c$

and :

$$\frac{d}{dt}(u-2c) = 0$$
 on the curves C- of equation $\frac{dx}{dt} = u - c$

On the curves C_+ the quantity u + 2c is invariant. On the curves C_- the quantity u - 2c is invariant.

The curves C_+ and C_- are called the characteristics. The values u + 2c and u - 2c are called the Riemann invariants. Existence of real characteristic curves places the Saint-Venant equations in the family of hyperbolic differential equations.

The characteristics are represented in a diagram (x,t). Two cases are considered (see following diagrams) according to the sign of u - c, u being chosen positive.



if u < c:	the flow is called fluvial or tranquil (also called streaming) the waves travel faster than the current the flow at any one point depends on conditions at upstream and downstream boundaries.	
if u > c:	the flow is called torrential (also called rapid or shooting)	

- the waves travel slower than the current the flow at any one point depends only on the upstream boundary conditions; it does not depend on downstream boundary conditions
- if u = c: the flow is called critical

The boundary conditions therefore change depending on the type of flow (tranquil, critical, or torrential). This phenomenon is very important in numerical modeling and it leads to the failure of many models.

The type of flow (tranquil or torrential) is given by Froude Number:

$$F = \frac{u}{c}$$
 in dimension 1
 $F = \frac{\sqrt{u^2 + v^2}}{c}$ in dimension 2

 $F \in [0, + \infty]$

if	F < 1:	the flow is tranquil
if	F = 1:	the flow is critical
if	F > 1:	the flow is torrential

II.2.7 Concepts of hydraulic jumps

The Saint-Venant equations, by their compressible nature, permit discontinuous solutions called hydraulic jumps.

II.2.7.1 Stationary hydraulic jumps, conjugate relations

Let us consider only dimension 1 and a steady flow in a channel of constant width and flat bottom. In the absence of source terms, of diffusion, and of friction, the Saint-Venant equations express:

- The conservation of discharge per unit width Q = h u
- The balance of the momentum, conservation of : $hu^2 + g\frac{h^2}{2}$

Let us assume the existence of a discontinuity, with values of h_1 and u_1 upstream, and h_2 and u_2 downstream. We have to verify:

$$h_1 \ u_1 = h_2 \ u_2 = Q$$

 $h_1 \ u_1^2 + g \ \frac{h_1^2}{2} = h_2 \ u_2^2 + g \ \frac{h_2^2}{2}$

An obvious solution is $h_1 = h_2$ and $u_1 = u_2$.

Replacing in the second equation: u_1 by Q/h_1 and u_2 by Q/h_2 , we get:

$$h_1 h_2 (h_1 + h_2) = \frac{2 Q^2}{g}$$

This identity is the conjugate relation (or relation of the jump) with the depths. It depends on the discharge. It can also be written as:

$$h_1 h_2 (h_1 + h_2) = 2 h_c^3$$

where h_c is the critical depth, such that: $h_c^3 = \frac{Q^2}{g}$

In an analogous way, the conjugate relation with velocities:

$$\frac{u_1^2 u_2^2}{u_1 + u_2} = \frac{g Q}{2} = \frac{u_c^3}{2}$$

 u_c is the velocity of critical flow: $u_c = \sqrt{g h_c}$

The conjugate relation with Froude number is more interesting because it is independent of all other parameters. In fact, starting from:

g
$$h_1^3 = \frac{Q^2}{F_1^2}$$
 and g $h_2^3 = \frac{Q^2}{F_2^2}$, the relation of the depths gives:

$$F_1^{2/3} F_2^{2/3} = \frac{1}{2} \left(\frac{1}{F_1^{2/3}} + \frac{1}{F_2^{2/3}} \right)$$

or :

$$F_2^2 = \frac{(1 + \sqrt{1 + 8F_1^2})^3}{64F_1^4}$$

 F_1 and F_2 play symmetrical roles in these formulas. When $F_1 > 1$, $F_2 < 1$ and vice versa: This signifies that a stationary hydraulic jump is a transition point between tranquil flow and torrential flow. The velocity and depth can be found from the preceding formula since:

$$h_2^3 = \frac{Q^2}{g F_2^2}$$
 et $U_2^3 = g Q F_2^2$

The crossing of a hydraulic jump is characterised by a head loss. The expression for the head is $H = h + \frac{u^2}{2g}$, expressed in units of depth. Between the upstream and downstream points of a hydraulic jump, the difference is:

$$\Delta H = H_1 - H_2 = h_1 + \frac{u_1^2}{2 g} - h_2 - \frac{u_2^2}{2 g}$$
$$= h_1 + \frac{Q^2}{2g h_1^2} - h_2 - \frac{Q^2}{2g h_2^2}$$

With the help of the conjugate relation with the depths, we easily arrive at:

$$\Delta H = \frac{(h_2 - h_1)^3}{4 h_1 h_2}$$

This value is necessarily positive because it corresponds to a head loss. The upstream therefore has to be a torrential flow and the downstream a fluvial flow.



Assume that the hydraulic jump moves at a velocity W. We can then consider a frame of reference in translation at velocity W that moves with the hydraulic jump. The flow in this frame of reference is steady and the Saint-Venant equations still hold good.

Conservation of discharge and of the momentum on either side of the hydraulic jump are written as:

$$\mathbf{h}_1 \left(\, \mathbf{u}_1 - \mathbf{W} \, \right) = \mathbf{h}_2 \left(\, \mathbf{u}_2 - \mathbf{W} \, \right)$$

$$h_1 (u_1 - W)^2 + g \frac{h_1^2}{2} = h_2 (u_2 - W)^2 + g \frac{h_2^2}{2}$$

From the first equation we can derive:

$$W = \frac{h_1 u_1 - h_2 u_2}{h_1 - h_2}$$

This gives the velocity of the hydraulic jump in terms of the characteristics of upstream and downstream conditions.

Note: W does not necessarily have the same direction as u_1 and u_2 , for example in the case of a tidal bore.

II.2.8 Source terms and volumic forces

We will now cover in detail the terms called F_x and F_y so far, i.e., volumic forces other than pressure and weight. Some of these forces, such as the influence of wind, are negligible in dimension 3 but appear in the Saint-Venant equations as source terms applied to the entire water mass. This is because the equations represent a vertical average.

We will cover the following:

- the bottom friction
- the Coriolis force
- the influence of the wind
- atmospheric pressure
- sources of momentum

II.2.8.1 Friction on the bottom

The key words concerning friction at the bottom are:

«LAW OF BOTTOM FRICTION» «FRICTION COEFFICIENT» When the coefficient varies, it is given by the subroutine STRCHE.

Before starting on the actual laws of friction, lets us recall the linear law.

Linear law:

This law is rarely applied because it does not represent reality: the friction is in reality a quadratic function of velocity. It can, however, be used to model other phenomena. Its form is:

 $F_{x} = -b u$ $F_{x} = -b v$

Where b is the friction coefficient expressed in s⁻¹.

The true laws of friction:

In dimension 1, the constraint due to friction of a flow on a flat plane that is parallel to the flow is expressed by the following formula: $\tau = \frac{1}{2} \rho Cf u^2$ (units: kg m⁻¹ s⁻²).

where ρ density of the liquid (kg/m³)
Cf friction coefficient (dimensionless)
u velocity of the flow (m/s)

The constraint is in the direction of the current and in dimension 2 therefore becomes:

$$\vec{\tau} = \frac{1}{2} \rho \operatorname{Cf} |\mathbf{u}| \vec{\mathbf{u}}$$

The Cf coefficient is rarely used and it is traditionally replaced by other coefficients among which the most common is the coefficient of ChŽzy (or ChŽzy's factor), denoted by C. C and Cf are linked by the formula:

$$C = \sqrt{\frac{2 g}{C f}} \quad (units : m^{1/2} s^{-1})$$

The friction at the bottom in our case will equal $-\frac{1}{\rho h}\vec{\tau} \cdot \vec{n}_{fond}$. Keeping in mind that the normal vector at the bottom has the norm: $\sqrt{1 + (\frac{\partial Zf}{\partial x})^2 + (\frac{\partial Zf}{\partial y})^2}$ which is the reciprocal of the cosine of the steepest slope at a point that we shall call cos(α), we arrive at ChŽzy's formula for

the friction force at the bottom (to be added to the momentum equation in the non-conservative form):

$$\vec{F} = -\frac{1}{\cos(\alpha)} \frac{g}{h C^2} |u| \vec{u}$$

The friction law of ChŽzy was established for a uniform flow but is applied to all types of flow. It is finally expressed thus in our equations:

<u>ChŽzy's law :</u>

$$F_x = -\frac{1}{\cos(\alpha)} \frac{g}{h C^2} u \sqrt{u^2 + v^2}$$

C: Chezy's coefficient

$$F_{y} = -\frac{1}{\cos(\alpha)} \frac{g}{h C^{2}} v \sqrt{u^{2} + v^{2}}$$

The empirical law of Manning-Strickler defines the value of the coefficient of Chezy:

 $C = K R_H^{1/6}$ where K (m^{1/3} s⁻¹) is the Strickler coefficient and R_H is the hydraulic radius. An approximation here is that R_H = h, which in principle is only true for very wide channels.

K = 20 to 40 for natural river beds = 50 to 90 for a concrete-lined channel

This leads to a new law, called Strickler's formula.

Law of Strickler:

 $F_x = -\frac{1}{\cos(\alpha)} \frac{g}{h^{4/3} K^2} u \sqrt{u^2 + v^2}$

K : Strickler's coefficient.

$$F_{y} = -\frac{1}{\cos(\alpha)} \frac{g}{h^{4/3} K^{2}} v \sqrt{u^{2} + v^{2}}$$

In view of the approximation of the value of the hydraulic radius, a comparison with the results in dimension 1 is tricky. A calibration done in dimension 1 will not always be valid for dimension 2.

Manning's formula used mainly in anglo-saxon countries is a simple variation of Strickler's formula, with a coefficient m that is the reciprocal of K. See reference [15] for advice on choosing coefficients.

Manning's Law:

$$\begin{split} F_x &= -\frac{1}{\cos(\alpha)} \frac{g m^2}{h^{4/3}} u \sqrt{u^2 + v^2} \\ F_y &= -\frac{1}{\cos(\alpha)} \frac{g m^2}{h^{4/3}} v \sqrt{u^2 + v^2} \end{split}$$
m : coefficient of Manning.

Note: because of its use in anglo-saxon countries, the Manning coefficient is sometimes given in imperial units.

Nikuradse's Law:

According to Nikuradse's law, the coefficient of ChŽzy is obtained from the formula:

C = 7.83 Log (12
$$\frac{h}{k_s}$$
)

where h is the depth and k_s is grain size at the bottom.

Depending on the law finally selected, the coefficient of friction entered by the user will be: b, C, K, m, or k_s .

II.2.8.2 Coriolis force

The key words relating to the Coriolis force are:

«CORIOLIS» «CORIOLIS COEFFICIENT»

The Coriolis force is equal to:

$$-2\vec{\omega}\wedge\vec{u}$$

where $\vec{\omega}$ is the rotation vector of the earth and \vec{u} the velocity of the fluid.

At a point of latitude λ , we get:

$$\begin{split} F_x &= 2 \omega v \sin \lambda = f v \\ F_v &= -2 \omega u \sin \lambda = -f u \end{split}$$

In small domains, the coefficient $f = 2 \omega \sin \lambda$ is considered a constant: it is the «CORIOLIS COEFFICIENT» asked of the user. In the Mercator projection, this parameter is variable in space and it is calculated by the software (this process is activated by the key word «SPHERICAL COORDINATES».

The angular velocity of the earth is: $\omega = 7.292 \text{ x } 10^{-5} \text{ rd/s}$

(There are π radians in a sidereal day, equal to 0.997270 days of 24 hours, that is, 86164 s).

For a latitude of 48;, we have then $f = 1.083 \times 10^{-4} N m^{-1} s$.

II.2.8.3 Influence of wind:

The key words concerning the influence of wind are:

«WIND» «WIND VELOCITY ALONG X» «WIND VELOCITY ALONG Y» «COEFFICIENT OF WIND INFLUENCE»

The METEO subroutine allows the programming of the acquisition of wind data if it is variable in time and space.

Analogous to the analysis of friction at the bottom, the resistance of the wind takes the following form (we neglect the slope of the free surface):

$$F_{x} = \frac{1}{h} \frac{\rho_{air}}{\rho} a_{vent} U_{vent} \sqrt{U_{vent}^{2} + V_{vent}^{2}}$$
$$F_{y} = \frac{1}{h} \frac{\rho_{air}}{\rho} a_{vent} V_{vent} \sqrt{U_{vent}^{2} + V_{vent}^{2}}$$

where

 a_{vent} is a wind-resistance coefficient and U_{vent} , V_{vent} the components of the wind velocity in m/s (strictly speaking, it is the wind velocity with respect to the fluid).

 ρ_{air}/ρ is the ratio of the air and water densities.

The coefficient a_{vent} hides complex phenomena. In fact, the influence of the wind depends on the smoothness (or, lack of it) of the free surface and the distance over which it acts (called the «fetch»).

Value of a_{vent} can be obtained from many different formulas.

This is the formula used by the Institute of Oceanographic Sciences (United Kingdom):

 $\begin{array}{lll} \text{if } \left| \vec{U}_{vent} \right| < 5 \text{ m/s} & a_{vent} &= 0,565 \ 10^{-3} \\ \\ \text{if } 5 &< \left| \vec{U}_{vent} \right| < 19,22 \text{ m/s} & a_{vent} &= (-0,12 + 0,137 \ \left| \vec{U}_{vent} \right| \) \ 10^{-3} \\ \\ \text{if } \left| \vec{U}_{vent} \right| &> 19,22 \text{ m/s} & a_{vent} &= 2,513 \ 10^{-3} \\ \end{array}$

Important note !

The parameter «COEFFICIENT OF WIND INFLUENCE» asked for by TELEMAC-2D is:

 ρ_{air} / $\rho~a_{vent}$ and not a_{vent}

 ρ_{air} is approximately 1.023 kg/m³ and ρ is 1000 kg/m³.

II.2.8.4 Atmospheric pressure:

The key word to allow the inclusion of the effects of atmospheric pressure is:

«AIR PRESSURE»

The METEO subroutine allows the programming of the acquisition of atmospheric pressure data.

When we take into account the atmospheric pressure P_a (unit: Pascal = kg m¹ s⁻²), it simply adds to the hydrostatic pressure. So we simply add: $-\frac{1}{\rho} \overrightarrow{\text{grad}}(P_a)$ to the right hand side of the momentum equation. The pressure term is sometimes added to the free surface that itself is represented by its gradient. The gradient of atmospheric pressure should not be included in the case of tidal flats where it will be a parasitic driving term.

II.2.8.5 Sources of momentum

The key words concerning the sources are:

«WATER DISCHARGE OF SOURCES» «ABSCISSAE OF SOURCES» «ORDINATES OF SOURCES» «SOURCE VELOCITY ALONG X» «SOURCE VELOCITY ALONG Y»

As already seen in paragraph II.2.5.2, the discharge sources induce momentum sources equal to:

$$F_x = \frac{1}{h} (u_{sce} - u)$$
 Sce

$F_y = \frac{1}{h} (v_{sce} - v)$ Sce

u, v, u_{sce} , v_{sce} , and Sce are expressed in m/s.

Important note:

When the source velocities are not provided by the user, these terms are not added to the equations. We therefore implicitly assume that the velocity at the exit is that of the current. This choice has been made to retain compatibility with earlier releases of TELEMAC-2D.
II.2.9 Saint-Venant equations in Mercator projection:

The key words concerning the Mercator projection are:

«SPHERICAL COORDINATES » «LATITUDE OF ORIGIN POINT»

The latitude calculations from the Mercator coordinates are done by the LATITU subroutine which can be modified.

The calculations of storm surges are obtained by solving the Saint-Venant equations. The physics of these phenomena effectively ensures that vertical acceleration can be ignored and that the flow is quasi-horizontal. Nevertheless, to be able to model the phenomenon satisfactorily, the calculation domain has to be sufficiently vast. The currents and water elevations of storm surges are due to wind and atmospheric pressure action over large expanses of the ocean.

The large maritime regions are represented on maps using the Mercator projection of portions of the earth. Since the region to be studied is a Mercator projection, we have to determine what modifications to make to the Saint-Venant equations.

Mercator projection:

The Mercator projection belongs to the family of conformal mappings that conserves local angles but not the distances. In other words, the scale of the map is not the same at every point but at any given point, the scale is identical in all directions. The region to be represented is a flat plane on the surface of a sphere whose points are represented by their latitudes λ and their longitudes φ . In all that follows, we shall assume that the latitude and longitude are expressed in radians, assumed positive respectively for the northern hemisphere and east of the Greenwich meridian:



Starting from the sphere we arrive at a plane called the plane of Mercator with the help of a projection that has a point M of the sphere associated with a point M_p of a cylinder tangent to the earth at the equator (see sketch below).



Thus the coordinates x_p and y_p on this plane are derived from the geographic coordinates by:

However, the coordinates in this frame of reference do not correspond to a conformal mapping because:

- for dx, an East-West displacement on the surface of the earth, we get a displacement dx_p in this frame such that: $dx = \cos \lambda dx_p$.
- whereas for dy, a North-South displacement, we get a displacement dy_p such that: $dy = cos^2 \lambda dy_p$.

The distortion in the North-South direction which is more important than in the East-West direction makes it a mapping that is not conformal. We regain the status of a conformal mapping by a simple change in scale in the y direction. The new ordinate Y derives from y_p , and from λ , by :

$$dY = \cos \lambda \ dy_p = \frac{R}{\cos \lambda} \ d\lambda$$

At the same time, we include the possibility of the user changing the origin at a point (φ_0, λ_0). λ_0 corresponds to the key word «LATITUDE OF ORIGIN POINT».

Considering the x-axis, we arrive immediately at:

$$\mathbf{X} = \mathbf{R} \; (\mathbf{\phi} - \mathbf{\phi}_0)$$

Considering the y-axis, we arrive by integration at:

$$\mathbf{Y} = \mathbf{R} \left[\ln \left[tg \left(\frac{\lambda}{2} + \frac{\pi}{4} \right) \right] - \ln \left[tg \left(\frac{\lambda_0}{2} + \frac{\pi}{4} \right) \right] \right]$$

Since these X and Y coordinates are supplied to TELEMAC-2D, we should be able to express the Saint-Venant equations in these coordinates. X and Y being functions of spherical coordinates, it will be useful to begin by expressing our Saint-Venant equations in spherical coordinates. To do so we make use of the expressions of divergence and gradient operators in these coordinates.

$$\operatorname{div}(\vec{F}) = \frac{1}{R \cos \lambda} \frac{\partial F_{\varphi}}{\partial \varphi} + \frac{1}{R \cos \lambda} \frac{\partial (\cos \lambda F_{\lambda})}{\partial \lambda}$$
$$\overrightarrow{\operatorname{grad}}(f) = \frac{1}{R \cos \lambda} \frac{\partial f}{\partial \varphi} \vec{e}_{\varphi} + \frac{1}{R} \frac{\partial f}{\partial \lambda} \vec{e}_{\lambda}$$

Given the nature of our physical variables and of the projection, the scalar and vector values (elevation, velocity) are preserved . For example, we have $F_{\phi}=F_x\,$ et $\,F_{\lambda}=F_y$.

The derivatives of the functions with respect to φ and λ are then replaced by the derivatives in X and Y, the Mercator coordinates:

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial X} \frac{\partial X}{\partial \phi} + \frac{\partial f}{\partial Y} \frac{\partial Y}{\partial \phi} = R \frac{\partial f}{\partial X}$$
$$\frac{\partial f}{\partial \lambda} = \frac{\partial f}{\partial X} \frac{\partial X}{\partial \lambda} + \frac{\partial f}{\partial Y} \frac{\partial Y}{\partial \lambda} = \frac{R}{\cos \lambda} \frac{\partial f}{\partial Y}$$

We derive from these new expressions for the divergence and for the gradient:

$$\operatorname{div}(\vec{F}) = \frac{1}{\cos\lambda} \frac{\partial F_x}{\partial X} + \frac{1}{\cos\lambda} \frac{\partial F_y}{\partial Y} - \frac{\operatorname{tg}\lambda}{R} F_y = \frac{1}{\cos^2\lambda} \frac{\partial}{\partial X} (\cos\lambda F_x) + \frac{1}{\cos^2\lambda} \frac{\partial}{\partial Y} (\cos\lambda F_y)$$
$$\overrightarrow{\operatorname{grad}}(f) = \frac{1}{\cos\lambda} \left(\frac{\partial f}{\partial X} \right)$$

With the Mercator coordinates, the continuity equation that has to be solved is, in nonconservative form:

$$\frac{\partial h}{\partial t} + \frac{\vec{u}}{\cos \lambda} \cdot \overrightarrow{\text{grad}}(h) + \frac{h}{\cos \lambda} \operatorname{div}(\vec{u}) - \frac{hv}{R} \operatorname{tg} \lambda = 0$$

Until release 2.3 of TELEMAC-2D, it was this form of the equation that was used (see [3]), with the calculations of $\frac{\vec{u}}{\cos \lambda}$ and $\frac{h}{\cos \lambda}$ considered linear functions. Unfortunately, this form does not allow the strict conservation of mass. For this, the function $\lambda(x,y)$ would have to be included in the variational formulation and that is not possible since neither $\frac{1}{\cos \lambda}$ nor tg λ are polynomials.

The continuity equation can also be written in this form:

$$\frac{\partial(h\cos^2\lambda)}{\partial t} + \operatorname{div}(\cos\lambda h\vec{u}) = 0$$
 where the divergence is the cartesian operator.

We then establish that the integration over the Mercator domain of this equation (see chapter II.4.3 on variational formulation) will give the following terms:

$$\frac{\partial}{\partial t} \int_{\Omega} h \cos^2 \lambda \, dX \, dY \quad \text{and} \quad \int_{\Omega} \text{div}(\cos \lambda h \, \vec{u}) \, dX \, dY = \int_{\Gamma} h \, \vec{u} \, . \, \vec{n} \cos \lambda \, d\Gamma$$

In these terms, each dimension dX or dY is multiplied by $\cos \lambda$, which restores a constant local scale. The values obtained by calcutating volumes or fluxes are therefore real. In the same way, the gradient terms give real values for the slopes. Since in all operations, the coordinates appear multiplied by $\cos \lambda$, the idea is to do this multiplication once, before the beginning of the calculations. This can be formalised by a choice of a latitude λ constant for each element. The advantage over the solution obtained from previous releases is that the equations to be solved are strictly identical to cartesian equations (this is, of course, due to the local property of our operators and is not a general property). This choice of a constant latitude for each element is, of course, an approximation, as were the choice of linear functions $\frac{\vec{u}}{\cos \lambda}$ and $\frac{h}{\cos \lambda}$. But it permits us to retain a rigourous finite element formalism, since the approximation is, in some part, transferred to the coordinates.

Henceforth, the solution adopted by TELEMAC-2D is as follows:

Before any calculation, the coordinates of each point on the grid are multiplied by $\cos \lambda$. The Mercator coordinates are also retained but will be used only for graphical output.

Once the coordinates are so modified, the Mercator projection is automatically taken into account and no more modifications appear in the Saint-Venant equations. All the calculations of mass and flux are consistent and provide real values.

This procedure differs slightly from that of a choice of a constant latitude for each element, but is of the same order of approximation. This approximation has, on the other hand, an advantage in being able to be evaluated while trying to find the form the Saint-Venant equations will take with coordinates X' and Y' such that:

 $X' = X \cos \lambda$ and $Y' = Y \cos \lambda$.

The difference at the continuous level between the equations leads to the appearance of neglected terms of the order of 1/R that take into account the variations of latitude along the length of an element.

Important note:

In the case of the TELEMAC-2D option «SPHERICAL COORDINATES», the X and Y coordinates supplied to TELEMAC-2D differ only by a constant scale coefficient K from the coordinates x_c and y_c that are found on maritime charts:

$$X - X_0 = K x_c$$

$$Y - Y_0 = K y_c$$

with:

 (X_0, Y_0) : grid coordinates of the point of origin in the map.

The transition of a measured distance d on the ground to a distance D on the grid is obtained by a simple change of scale. This change of scale depends only on the latitude since the mapping being used is conformal. We therefore have:

$$D = \frac{d}{\cos \lambda}$$

Since each maritime chart provides the scale E at a given latitude λ_c , we derive the transition coefficient K to go from the chart to the grid:

$$K = \frac{1}{E \cos \lambda_c}$$

Remarks :

- The «SPHERICAL COORDINATES» option of TELEMAC-2D that takes into account the coordinates of Mercator is actually a misnomer. In fact, the actual Saint-Venant equations in spherical coordinates are obtained from the Navier-Stokes spherical equations that are averaged over polar coordinates. Some supplementary terms, such as the centrifugal force, are obtained in this manner.

- It would be possible with TELEMAC-2D to solve equations in a grid representing the integrality of the terrestrial sphere. In fact, we only use the local coordinates of each element. The consistency between the coordinates of different elements is not necessary (unless we use the characteristics method). With this approach, only 2D software drawing packages would encounter difficulties in outputting graphical results.

II.2.10 Spatial variations of density:

The key words for this option are:

«EFFECTS OF DENSITY»

«MEAN TEMPERATURE»

There are also the key words associated with the tracer because they also represent the salinity.

II.2.10.1 Momentum equations:

Since the Saint-Venant equations have been derived with an average over the vertical, they cannot represent the stratification resulting from the vertical variations of density, itself a function of salinity or of the temperature. It is nevertheless possible to take into account the horizontal variations in density. If we denote S as the salinity and θ the temperature, ρ is a function $\rho(S, \theta)$, where S and θ are functions of x, y, and t. These variations cause two effects:

- Dilatation of water.
- Differential effects of gravity.

The dilatation of water is a secondary effect which will be ignored. The differential effects of gravity due to variations in salinity, on the other hand, are evident in estuaries and have sometimes to be considered. To do this we use Boussinesq's hypothesis which consists of taking into account the variations in salinity only in terms of gravity. In all the terms of the equations we replace ρ by the reference ρ_0 , except in the pressure term, which in the Navier-Stokes equations (conservative with hydrostatic pressure), then becomes: $-\frac{1}{\rho_0} \operatorname{grad} \left[\rho g (Z - z) \right]$. When we integrate this term between the bottom and the surface while considering the variations of ρ over x and y, we get:

* A term
$$-\frac{\rho}{\rho_0} g h \overrightarrow{\text{grad}}(Z)$$
 that becomes $-\frac{\rho}{\rho_0} g \overrightarrow{\text{grad}}(Z)$ in the non-conservative Saint-

Venant equations (see paragraph II.2.5.2) and replaces the term - $g \operatorname{grad} (Z)$.

 $-\frac{\rho}{\rho_0}$ g grad (Z) is called «barotropic pressure gradient».

* A term $-\frac{g}{\rho_0} \int_{z_f}^{z} (Z-z) \overrightarrow{\text{grad}}(\rho) dz = -\frac{g}{\rho_0} \frac{h^2}{2} \overrightarrow{\text{grad}}(\rho)$ that in non-conservative equations becomes: $-\frac{g}{\rho_0} \frac{h}{2} \overrightarrow{\text{grad}}(\rho)$.

 $-\frac{g}{\rho_0}\frac{h}{2}\overrightarrow{\text{grad}}(\rho)$ is called «baroclinic pressure gradient».

II.2.10.2 Salinity equation:

To calculate the density at each point of the domain at each instant, the salinity has to be known and the density calculated from it. Therefore the tracer is considered the salinity S, expressed in kg/m^3 , subject to the advection, to the diffusion, and the effects of boundary conditions.

The variations in ρ are then given by this formula (see reference [14]):

$$\rho \ = \ \rho_{ref} \left[\ 1 - (\ 7(\theta - \theta_{ref})^2 - 750 \ S \) \ 10^{-6} \ \right] \quad for \ 0; C < \theta < 40; C \ and \ 0 \ g/l < S < 42 \ g/l$$

The «WATER DENSITY» parameter entered by the user is in this case recalculated.

 θ_{ref} is the reference temperature of 4 iC and ρ_{ref} the reference density at this temperature when the salinity is nil: $\rho_{ref} = 999.972 \text{ kg/m}^3$.

As we are ignoring the variations in temperature, we have to select an average temperature θ for the calculations. The value of ρ_0 of the barotropic and baroclinic pressure gradients is then:

 $\rho_0 = \rho_{ref} [1 - 7 \ 10^{-6} (\theta - \theta_{ref})^2]$

We have then $\rho = \rho_0 + 0.749979$ S (ρ will then formally have the same discretisation as S).

The parameter that the user is asked to enter is average temperature θ (key word «AVERAGE TEMPERATURE»). In TELEMAC-2D, the density effects are dealt with in the form of an additional term in the momentum equation and equal to:

 $-\frac{g}{\rho_0} \left[(\rho - \rho_0) \overrightarrow{\text{grad}}(Z) + \frac{h}{2} \overrightarrow{\text{grad}}(\rho - \rho_0) \right]$

II.2.11 Boundary conditions:

In the TELEMAC-2D code, a distinction is made between «physical» boundary conditions and «technical» boundary conditions. The latter are algorithmic representation of the former. The «physical» boundary conditions which we shall explore now, are those that are provided by the user. The «technical» boundary conditions will be covered in chapter III because they are closely linked to the model used.

Physically, we have first to distinguish between the two types of boundaries of the domain of calculations: the liquid boundaries and the solid boundaries.

Solid boundaries:

With this sort of boundary, there exists an impermeability condition: no discharge can take place across a solid boundary.

There is also the friction. The friction coefficient is either provided by the user or determined by the turbulence model. The coefficient «a» (aubor in the programme) is used to impose the following conditions:

$$\frac{\partial U}{\partial n} = a U$$
 and $\frac{\partial V}{\partial n} = a V$

These two conditions are stronger than the condition of ordinary friction:

$$\frac{\partial(\vec{U}.\vec{t})}{\partial n} = a \vec{U}.\vec{t}$$

but are compatible with the form chosen for the diffusion terms.

We shall see in chapter III that the variational formulation in finite elements entails these two conditions: impermeability and friction. In particular, no condition on the depth is imposed in the case of a solid boundary.

Liquid boundaries:

This type of a boundary is trickier to deal with because it supposes the existence of fluid domain that does not form part of the calculation domain but one that can nevertheless influence it. This influence has to be translated into boundary conditions.

Scrutiny of the Saint-Venant equations without a diffusion term and in dimension 1 using the characteristics method (paragraph II.2.6) leads to four distinct types of liquid boundaries:

* Entry with torrential flow

- * Entry with tranquil flow
- * Exit with torrential flow
- * Exit with tranquil flow

An inflow and an outflow are normally represented by the sign of the scalar product of the velocity vector and the outward normal vector (Inflow: negative product, Outflow: positive product). The contrary has been retained however for the key-words and the output listing, as the users generally think that a positive flux logically corresponds to an inflow.

The torrential and tranquil flows are distinguished by Froude's number:

$$F = \frac{u}{\sqrt{gh}}$$

This number expresses the ratio between the speed of the current and the celerity c of long waves. If F>1, the flow is torrential, if F<1, it is tranquil (see paragraph II.2.6).

According to the theory of characteristics, the boundary conditions are (in dimension 1);

* data consisting of two pieces of information for a torrential inflow.

- * data consisting of one piece of information for a tranquil inflow.
- * no conditions for a torrential outflow.
- * data consisting of one piece of information for a tranquil outflow.

Apart from the complications arising in dimension 2 due to the fact that the velocity is a vector, this theory is difficult to follow rigourously. In fact, the separation during the solving of the equations into two fractional steps introduces intermediate boundary conditions that do not correspond to reality. Thus the advection step for a liquid boundary with an inflow requires the data of all the advected variables at the inflow.

An additional difficulty is that the «inflow» or «outflow» characteristic of a liquid boundary can vary with time (for example, in the case of a boundary in an open sea with tides).

The solution selected in TELEMAC is based on the study done with the characteristics method. It is as follows:

We ask the user to indicate for each of the principal variables of the code (variables h, u, and v) if it is prescribed or free. This has to be indicated for each point of a liquid boundary. This permits us to arrange the different types of boundary conditions in a modular fashion adapted to the four types of boundaries mentioned above:

* Velocity and depth prescribed (torrential inflow)

- * Velocity prescribed and free depth (fluvial inflow)
- * Free velocity and depth (torrential outflow)
- * Free velocitiy and depth prescribed (fluvial outflow)

Prescribed flowrate conditions:

Very often, for example in the case of a river, the exact velocity profile over a section is not known in advance. But it is nevertheless necessary for the upstream side of a domain of calculations. The only data that is usually available is the general discharge. We then impose a velocity profile (for example, a constant or a logarithmic profile). And by the simple use of the rule of three, the exact numerical value of the prescribed velocities is obtained to get the required flowrate. This process is applicable even in the case of different inflows at different points and even if the prescribed flowrate is a function of the water depth. However, this latter case has never yet been tested.

Incident-wave conditions:

This condition lets enter (and leave) a sinusoidal wave through open boundaries of a domain. It is written, for a wave of frequency ω :

$$c\vec{u}.\vec{n} = gZ^n - g(1 - \vec{k}.\vec{n})\hat{S}$$

where:

c: celerity of waves.

 \vec{n} : unitary vector normal to the domain boundary pointing outward.

Zⁿ: elevation of the free surface obtained in the previous instant.

 \vec{k} : unitary wave vector of the incident signal, of norm $\frac{\omega}{c}$.

 $\widehat{S}=A(M)\cos\left(\omega t$ - $\phi(M)\right)$: incident sinusoidal signal homogenous at a depth.

This formula originates from the characteristics theory and the Riemann invariants. In fact, we write that in tranquil flow, the Riemann invariant coming from the outside of the domain corresponds to the characteristics of an incident wave. If there is no incident wave, this condition means in dimension 1 that the invariant u + 2c equals $2c_0$, where c_0 is the celerity on the outside of the domain boundary.

This boundary condition minimizes the reflection of waves exiting the domain in a direction that is normal to the boundary. The result depends, in fact, on the orientation of the boundary.

On the other hand, when a sinuso•dal wave enters a domain, its calculation requires the correct knowledge of the incident signal (amplitude, phase, frequency, wave vector) which is not the same as the measured signal since this latter includes the reflection of the waves.

In cases where we do not have complete information about the wave, we prefer to work with imposed values.

II.3 Tracer Equation:

The tracer represents a temperature or other passive physical quantity that does not change or modify the flow. This tracer can be used to study thermal dilution or contaminant flow. We must however bear in mind that the numerical value of the tracer is an average over the vertical, a fact that prevents the use of the tracer to represent thermal- or density-stratification. These phenomena can only be studied with three-dimensional codes such as TELEMAC-3D.

In reality, the evolution of the tracer depends on:

- * the current that carries it (advection),
- * the diffusion and the dispersion. When a tracer is concerned, the dispersion is modeled better than when it involves the momentum.
- * source and sink terms.

It is pointless to specify here the units the tracer is expressed in.

In dimension 3, before the averaging over the vertical, the tracer equation is:

 $\frac{\partial}{\partial t} (\rho T) + \text{div}(\rho T \vec{U} + \vec{q}) = F_{source}$

where F_{source} is the level of creation and \vec{q} is the flux caused by molecular diffusion or turbulence, of the form $\vec{q} = -\rho v_T \operatorname{grad}(T)$.

 ν_{T} is the coefficient of molecular or turbulent diffusion of the tracer.

The averaging over the vertical already done to obtain the Saint-Venant equations gives us the following tracer equation in conservative form in 2 dimension:

$$\frac{\partial}{\partial t}$$
 (hT) + div(hT \vec{u}) - div(h $v_T \operatorname{grad}(T)$) = T_{sce} Sce

 T_{sce} is the value of the tracer at the source (key word «VALUES OF THE TRACER AT THE SOURCES») and Sce the source of discharge, in m/s, linked to the source discharge Q_{sce} expressed in m³/s.

Here v_T takes into account the dispersion. The index T indicates that the diffusion applies to the tracer, it is no longer v_e .

This equation can be processed directly in this form, a treatment that would be more practical in assuring the conservation of the tracer. However, if suffers from the following drawbacks:

- The interesting variable T can only be obtained by dividing by the depth. This creates a problem of monotony of T, to say nothing of the case of tidal flats.

- If T is constant over the entire domain, the situation is not stable since it does not correspond to a trivial case of the equation: the variable hT is not constant if the depth varies.

For these reasons, we prefer the tracer equation in the non-conservative form, even if it too has its defects. The way it is obtained is detailed below, a process that has its importance because it should be possible to do the inverse operation at a discrete level if we would like to show the conservation of the tracer.

As a first step, the derivatives of the products are developed:

$$h\frac{\partial}{\partial t}(T) + T\frac{\partial}{\partial t}(h) + h \vec{u}. \vec{grad}(T) + T \operatorname{div}(h \vec{u}) - \operatorname{div}(h v_T \vec{grad}(T)) = T_{sce} \operatorname{Sce}$$

Then, we notice the presence of terms $T(\frac{\partial h}{\partial t} + \operatorname{div}(h \vec{u}))$ that, by using the continuity equation,

we replace by +T Sce where Sce is the source term of the continuity equation.

The remaining equation is then divided by h and we get, finally:

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} - \frac{1}{h} \operatorname{div}(h \ v_T \operatorname{grad} T) = \frac{(T_{sce} - T) \operatorname{Sce}}{h}$$

The right hand side of the equation is nil if the tracer value of the source is equal to the tracer value in the middle where the source arrives. This is not the general case for an inflow into the domain.

Let us recall that:

u and v are the components of the velocity field, h is the water depth, v_T is the coefficient of diffusion (laminar or turbulent) associated with the tracer, Sce is a source of flow expressed here in m/s.

The parameters that are asked of the user for each point where there is a source are:

Q _{sce}	: source flowrate in m^3/s .
T _{sce}	: source value of the tracer, in units the tracer is expressed in.
u_{sce} and v_{sce}	: components of the source velocity in m/s.

Boundary conditions:

The tracer boundary conditions are:

- prescribed value,
- free value (for an outflow, for example),

- flux condition.

The flux condition takes the form:

$$\frac{\partial T}{\partial n} = a T + b$$

where a and b are provided by the user (under the names atbor and btbor in TELEMAC-2D).

For a condition of free outflow and if we use the characteristics method, we can prescribe the result of the advection, ignoring in it the effect of the diffusion and source terms. In fact, the diffusion stage demands the form of the flux condition which would be, in this case, an artificial boundary condition. In particular, a nil flux condition is not equivalent to a condition of free outflow.

II.4 Turbulence Modelling:

II.4.1 Equations of the k-epsilon model:

The presence of a turbulence model is related to finding the average of the Navier-Stokes equations over time. This averaging introduces additional terms which form the Reynolds tensor, due to the energy transfers between the average movement and the eddies caused by the turbulence. The system of equation obtained is not solvable because it is not closed. This difficulty has therefore made it necessary to model the new terms more or less empirically and has given rise to turbulence models called closure models.

The closed system of equation can be got either by expressing Reynolds tensor as a function of the average values directly as algebraic relations (0 equation model) or by a combination of algebraic relations and a transport equation expressing the turbulence level (1 equation model), or as done here, by solving directly the transport equations for the kinetic energy and its rate of dissipation which constitutes the k- ε model (2 equations model).

k and ε are values averaged vertically which are formally defined as follows:

$$k = \frac{1}{h} \int_{z_{f}}^{z} \frac{1}{2} \overline{u'_{i}u'_{i}} dz \text{ (sum for } i=1 \text{ to } 3\text{)}.$$

$$\epsilon = \frac{1}{h} \int_{z_{f}}^{z} \frac{v}{2} \overline{\frac{\partial u'_{i}}{\partial x_{j}} \frac{\partial u'_{i}}{\partial x_{j}}} dz \text{ (sum for } i=1 \text{ to } 3 \text{ and for } j=1 \text{ to } 3\text{)}.$$

where u_i' represents temporal fluctuation of velocity and the horizontal bar corresponds to the average over time. The closing is done by the hypothesis of Boussinesq which expresses Reynolds tensor as a function of the velocity gradients:

$$\frac{R_{ij}}{\rho} = v_t \left(\frac{\partial U_i}{\partial x_i} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} k \, \delta_{ij} \qquad (i,j) \in \langle 1,2 \rangle$$

where R_{ij} is the Reynolds tensor. The last term of this relation which contains k is neglected (in Navier-Stokes equation this term is an integral part of pressure which is not possible here). According to Kolmogorov, the turbulent viscosity is expressed finally as:

$$\boldsymbol{n}_t = c_m \frac{k^2}{\boldsymbol{e}}$$

The turbulence model used in TELEMAC-2D is an extension of the classical model, put forward by Rastogi and Rodi [13], to the Saint-Venant equations. The vertical integration of Reynolds equations produces dissipation terms which do not exist in the classical model and which arise due to the non uniformity of the vertical profile of velocity.

The system to be solved is then of the form :

$$\frac{\partial k}{\partial t} + \vec{u}.\vec{grad}(k) = \frac{1}{h} \operatorname{div}(h \; \frac{v_t}{\sigma_k} \; \vec{grad}(k)) + P - \epsilon + P_{kv}$$
$$\frac{\partial \epsilon}{\partial t} + \vec{u}.\vec{grad}(\epsilon) = \frac{1}{h} \operatorname{div}(h \; \frac{v_t}{\sigma_\epsilon} \; \vec{grad}(\epsilon)) + \frac{\epsilon}{k} (c_{1\epsilon}P - c_{2\epsilon}\epsilon) + P_{\epsilon v}$$

The first terms on the right hand side are diffusion terms, whereas the production terms (always positive) are composed on one hand of horizontal gradient of velocity:

$$P = v_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} \qquad (i,j) \in (1,2)$$

and on the other hand of the terms of vertical shear :

$$P_{kv} = c_k \frac{u_*^3}{h} \quad \text{avec} : \ c_k = \frac{1}{\sqrt{c_f}}$$
$$P_{\varepsilon v} = c_{\varepsilon} \frac{u_*^4}{h^2} \quad \text{avec} : \ c_{\varepsilon} = 3.6 \frac{c_{2\varepsilon} \sqrt{c_{\mu}}}{c_{\mathrm{f}}^{3/4}}$$

where C_f is the coefficient of friction u_* the friction velocity at the bottom which, by defination, is equal to $\sqrt{t/r}$, where τ is the stress on the bottom. The friction velocity is equal to :

$$\mathbf{u}_* = \sqrt{\mathbf{C}_{\mathbf{f}}(\mathbf{u}^2 + \mathbf{v}^2)}$$

The constants of the k- ε model are based on classical test cases. The free decrease of turbulence enables finding a value of c_{2e} from well documented experimental data. The constants c_m and c_{1e} are determined from the data for a turbulent flow near a solid wall. The constants \mathbf{s}_k and \mathbf{s}_e have been "optimised", on the basis of the performance of the model in the two test cases. Finally, the values of constants c_k and c_e are determined from the normal flow in the centre of a straight channel. The set of constants retained for the k- ε model, which is the same for all configurations is given in the following table:

с _т	<i>c</i> _{1<i>e</i>}	<i>c</i> _{2<i>e</i>}	$oldsymbol{s}_k$	s _e
0.09	1.44	1.92	1.0	1.3

Constants for the k- ε model

Finally, the friction coefficient C_f is deduced from the law of friction chosen with the key word "LAW OF BOTTOM FRICTION".

1 : Linear formula (not an object here).



where C is ChŽzy's coefficient, K is Strickler's factor, m = 1/S is Manning's factor and k_s is the grain size for the Nikuradse's formula, with von Karman's constant k = 0.41. The last formula is valid only for rough bottoms and is based on a logarithmic vertical profile of velocity. The key word «FRICTION COEFFICIENT" represents any of the coefficients C, K, m and k_s depending on the choice of «LAW OF BOTTOM FRICTION».

II.4.2 Boundary conditions of the k-epsilon model:

Solid boundary :

To define the boundary conditions, we consider that there exists a local turbulence equilibrium at the solid boundary such that: Production (by shear stress on the boundary and the bottom) = Dissipation and the velocity profile is locally logarithmic. From these hypotheses, the value of ε at a distance δ (defined later) from the boundary and for a water depth *h* is given by:

$$\boldsymbol{e_d} = \frac{u_*^3}{\boldsymbol{kd}} + \frac{1}{\sqrt{c_f}} \frac{\tilde{u}_*^3}{h}$$

 $\mathbf{k} = 0.41$ is Karman's constant and u_* the friction velocity, which is yet to be determined. \tilde{u}_* is the bottom friction velocity. The kinetic energy is supposed to be constituted of two parts, one is the contribution from the boundary and the other from the bottom friction. The following expression is deduced:

$$k_{\mathbf{d}} = \frac{u_*^2}{\sqrt{c_{\mathbf{m}}}} + \frac{c_{2\mathbf{e}}}{c_{\mathbf{e}}c_f} \tilde{u}_*^2$$

with the constants defined above.

The distance from the boundary, d, will now be determined. We suppose simply that the half width of a transverse section D of any canal will be covered by at least one grid one point of which, inside the domain, is situated at a distance Δ from the boundary. The choice of the

distance from the boundary, d, is then fixed as the upper limit for the domain of validity of the logarithmic profile. This profile extends up to at least $\delta = 0.1$ D. Consequently it is deduced that:



The friction velocity on the boundary, \tilde{u}_* , is calculated according to the friction law chosen with the key word «TURBULENCE MODEL FOR SOLID BOUNDARIES».



For a smooth boundary the equation is solved by iteration, the initial value is so taken that $u_*d/n=100$, which amounts to a turbulence intensity of 6%. Generally 5 iterations are sufficient to converge to a stable solution. Finally, the lower limit used is $\frac{u*\delta}{v} > 30$, which gives a maximum turbulence intensity of 7.25%.

The key word «BOUNDARY ROUGHNESS» represents any of the factors k_s , C and K as per the choice for the key word «TURBULENCE MODEL FOR SOLID BOUNDARIES».

The k-epsilon model provides a value for the friction coefficient at the boundary, which is defined by the following relation:

$$n_t \frac{\P U_{tg}}{\P y} = -a \ U_{tg}^{n+1}$$
 avec: $U_{tg} = \sqrt{u^{n^2} + v^{n^2}}$

Still based on the same hypothesis of equilibrium between production and dissipation, the logarithmic relation between the friction velocity and \tilde{u}_* the velocity tangential to the boundary U_{tg} , provides the following relation :

$$\frac{\P U_{\text{tg}}}{\P y} = \frac{u_*}{ky} \quad \text{et } \boldsymbol{u}_{\text{t}} = \boldsymbol{k} y u_*$$

which enables the following expression for a :

$$a = -\frac{\tilde{u}_*^2}{U_{tg}}$$

Liquid Boundary :

We choose homogeneous Neumann conditions for an exit:

$$\frac{\P k}{\P n} = 0 \quad ; \quad \frac{\P \mathbf{e}}{\P n} = 0$$

where the vector \vec{n} is perpendicular to the boundary. As regards the entry conditions, we suppose that there exists an equilibrium between the production of kinetic energy by the bottom friction and its rate of dissipation. The equations of k and ε give for open boundary conditions:

$$\varepsilon = P_{kv} \quad ; \quad k = c_{2\varepsilon} \frac{P_{kv}^2}{P_{\varepsilon v}}$$

The values of k and ε are also used within the domain to initialise the calculations.

III. SOLUTION ALGORITHMS:

III.1 Brief description:

We will cover the methods used to give an idea of the set of solution schemes:

- The starting equations are the non conservative Saint-Venant equations in depth and velocity. If the method of characteristics is chosen, these equations are treated in 2 steps using the method of fractional steps.

- In the first step the advection terms corresponding to transport of the physical variables h, u, and v and in the alternative case k, ε and T are treated. The hyperbolic character of the equations justifies the use of the characteristics method.

- In the second stage the remaining equations are treated: propagation, diffusion, source terms, and the advection terms when there is no recourse to the characteristics method. This stage is solved by the finite element method; discretisation of time allows the elimination of non linearity of the equations. Variational formulations and discretisation of space transform the continuous equations into a linear discrete system where the values of h, u, v at the nodes are the unknown variables. This system is solved by an iterative conjugate gradient method.

III.2 Fractional step method :

This method is applied only when we choose the characteristics method.

As with space time is discretised and the unknowns are located at multiples of the given time step DT. Thus $t^n = t^0 + n$ DT and the derivative with respect to time of a function f is discretised as:

$$\frac{\partial f}{\partial t} = \frac{f^{n+1} - f^n}{DT}$$

 f^{n+1} is the function f at time f^{n+1} (new time step), f^n the function f at time f^n (previous time step).

This approximation is of the first order, unless we consider that the derivative is taken at time $t^n + DT/2$; in which case we obtain the second order. We try therefore to discretise the other terms of the equations at this same moment, which is not always possible especially with advection.

Note : this presentation of time derivatives is a finite difference approach. In Finite Elements,

<u>Note</u>: this presentation of time derivatives is a finite difference $a_{present}$ $f^{n+1} - f^n$ can be interpreted as the result of an integration by parts of the term $\int_{-\infty}^{t^{n+1}} \frac{\partial f}{\partial t} dt$ resulting

from a variational formulation which includes the variable time.

The general principles of solution are as follows:

Starting from an initial solution f^0 at time t^0 , the solution f^1 is obtained by successive iterations (time steps) by finding first f^1 then f^2 , etc.

Within the time step, the method of fractional steps consists of finding f^{n+1} , starting from f^{n} , and passing through intermediate steps which take only some of the terms of the equations into account. To be precise, in our case, we shall solve the equation :

$$\frac{\tilde{f} - f^n}{DT} + \text{ advection terms} = 0$$

then:

 $\frac{\underline{f^{n+1}} - \widetilde{f}}{DT} + other \ terms = 0$

where f represents h, u, v, k , ϵ or T.

We get back the starting equations from the sum of these two equations; however the approximation in time will be of order 1 because certain terms (advection) are not discretised at time $t^n + DT/2$.

This drawback of the order in time is compensated by the fact that each term may be treated with the technique that is the most appropriate for it. Though nothing prevents the use of second order approximation in the characteristic method, this possibility has not been explored in the TELEMAC framework. In practice, in this method, order 2 in time implies that there are two stages of convection (with two different time steps DT and 2 DT, and a modified advecting field

equal to $2\vec{u}^n - \vec{u}^{n-1}$) yielding the results \tilde{f}^1 and \tilde{f}^2 ; the combination $\frac{\frac{3}{2}f^{n+1} - 2\tilde{f}^1 + \frac{1}{2}\tilde{f}^2}{DT}$ gives the second order.

III.3 Advection step by the method of characteristics:

All or some of these equations can be solved (as per choice), between the time t^n and time t^{n+1} :

$$\frac{\widetilde{h} - h^{n}}{DT} + \overrightarrow{u} \cdot \overrightarrow{\text{grad}}(h) = 0$$

$$\frac{\widetilde{u} - u^{n}}{DT} + \overrightarrow{u} \cdot \overrightarrow{\text{grad}}(u) = 0$$

$$\frac{\widetilde{v} - v^{n}}{DT} + \overrightarrow{u} \cdot \overrightarrow{\text{grad}}(v) = 0$$

$$\widetilde{T} - \frac{T^{n}}{DT} + \overrightarrow{u} \cdot \overrightarrow{\text{grad}}(V) = 0$$

$$\widetilde{K} - \frac{k^{n}}{DT} + \overrightarrow{u} \cdot \overrightarrow{\text{grad}}(K) = 0$$

$$\widetilde{\epsilon} - \frac{\epsilon^{n}}{DT} + \overrightarrow{u} \cdot \overrightarrow{\text{grad}}(\epsilon) = 0$$

Each of these equations simply states that the total derivative of the functions h,u,v, etc... is zero along the trajectory known as characteristic. The quantities transported are therefore constant along these characteristics. The details of calculations of the characteristic curves are given later in paragraph III.9.1. The note on the treatment of the terms which normally figure on the right hand side should in particular be referred to.

III.4 Propagation - diffusion - source terms :

When the method of characteristics is not used, this stage solves the whole set of equations.

III.4.1 Discretisation in time :

As explained in the fractional step method the time derivative of a function f is of the form :

$$\frac{\partial f}{\partial t} = \frac{f^{n+1} - f^n}{DT}$$

DT is the key word «TIME STEP».

Solutions must be found for :

Case with prior advection:

 \widetilde{h} , \widetilde{u} and \widetilde{v} are the results of the advection of h, u and v.

$$\frac{h^{n+1} - \tilde{h}}{DT} + h \operatorname{div}(\vec{u}) = \operatorname{Sce}$$

$$\frac{u^{n+1} - \tilde{u}}{DT} = -g \frac{\partial Z}{\partial x} + F_x + \frac{1}{h} \operatorname{div}(h \ v_e \ \overline{\operatorname{grad}} \ u)$$

$$\frac{v^{n+1} - \tilde{v}}{DT} = -g \frac{\partial Z}{\partial y} + F_y + \frac{1}{h} \operatorname{div}(h \ v_e \ \overline{\operatorname{grad}} \ v)$$

Strictly the terms to the right of the = sign should be obtained from an integration over time along the characteristic curves (see the note of paragraph III.9.1)

Case with no prior advection :

$$\frac{h^{n+1} - h^n}{DT} + \vec{u} \cdot \vec{grad}(h) + h \operatorname{div}(\vec{u}) = \operatorname{Sce}$$

$$\frac{u^{n+1} - u^n}{DT} + \vec{u} \cdot \vec{grad}(u) = -g \frac{\partial Z}{\partial x} + F_x + \frac{1}{h} \operatorname{div}(h \ v_e \ \vec{grad} \ u)$$

$$\frac{v^{n+1} - v^n}{DT} + \vec{u} \cdot \vec{grad}(v) = -g \frac{\partial Z}{\partial y} + F_y + \frac{1}{h} \operatorname{div}(h \ v_e \ \vec{grad} \ v)$$

We have a selection of the above cases according to the option made.

There exist in these equations the terms h, u, v which are not clearly defined as to whether they correspond to time t^n or t^{n+1} . If we wish to consider the second order we could write $h = (h^n + h^{n+1})/2$ and similarly for u and v. This discretisation in time, called semi-implicitation, cannot be strictly applied. Fourier analysis of the semi-implicit method would yield an amplification coefficient equal to 1. Such methods though perfect in principle tend to diverge in practice. This is because Fourier analysis does not take into account non-linearity or the boundary conditions. A safety margin is provided by expressing a function f as:

 $f = \theta f^{n+1} + (1-\theta) f^n$, with $\theta > 0.5$ while always remaining close to this value.

We retain the description «semi implicit» even if θ is greater than 0.5.

Sub-iterations for non-linearity :

This semi-implicitation does not solve all problems. In fact the non-linear term $h \operatorname{div}(\vec{u})$ involves the product of the unknown variables h^{n+1} and u^{n+1} . We would obtain a non-linear system if no adjustments were made. Therefore the following formulae are adopted:

 $h \operatorname{div}(\vec{u}) = h_{prop} \operatorname{div}(\theta_{u} \vec{u^{n+1}} + (1 - \theta_{u}) \vec{u^{n}})$

where θ_u is the velocity implicitation coefficient (key word «IMPLICITATION FOR VELOCITY».

 h_{prop} is calculated by a sub-iteration, by successively solving for the same time step, by the formula:

 $h_{prop} = \theta_h h'^{n+1} + (1-\theta_h) h^n$. h'^{n+1} is an estimate of h^{n+1} ; it is equal to h^n when no subiterations are carried out else, with sub-iterations, it is equal to the value h^{n+1} determined in the previous sub-iteration. h_{prop} is called «propagation depth».

 θ_h is the depth implicitation coefficient (key word "IMPLICITATION FOR DEPTH").

Similarly, when the advection terms are not treated by the characteristics method, \vec{u} . $\vec{grad}(f)$ will be written as :

 \vec{u}_{conv} . $\vec{grad}(\theta_f f^{n+1} + (1-\theta_f) f^n)$ where \vec{u}_{conv} is the advection velocity given by: $\theta_u \vec{u}^{n+1} + (1-\theta_u) \vec{u}^n$, where \vec{u}^{n+1} itself is an approximation of \vec{u}^{n+1} obtained by sub-iteration.

Implicitation coefficients are in principle always greater than 0.5 (generally 0.6 yields good results).

Diffusion terms:

Though the diffusion terms were semi-implicit in TELEMAC 2D for a long time, a total implicitation, which is simpler and gives similar results, will henceforth be chosen. Therefore:

div
$$(\nu \overrightarrow{\text{grad}}(u)) = \text{div}(\nu \overrightarrow{\text{grad}}(u^{n+1}))$$

Implicitation coefficient for diffusion still forms part of the data in TELEMAC but is not taken into account.

Incremental formulation for depth:

The function Z (free surface) should be expressed in function of the elevation of the bottom and the water depth. Considering the semi-implicitation of h, we write :

$$-g \overrightarrow{\text{grad}}(Z) = -g \overrightarrow{\text{grad}}(\theta_h h^{n+1} + (1-\theta_h)h^n + Zf)$$

And also :

$$-g \overrightarrow{\text{grad}}(Z) = -g \theta_h \overrightarrow{\text{grad}}(h^{n+1} - h^n) - g \overrightarrow{\text{grad}}(Z^n)$$
 where $Z^n = h^n + Zf$.

In view of this expression and the continuity equation it is more elegant to replace the unknown h^{n+1} by the increment $h^{n+1} - h^n$. In the case of a still lake with irregular bottom, the term $-g \operatorname{grad}(Z^n)$ will be zero and the increment $h^{n+1} - h^n$ will also be zero. The conservation of the state of rest with irregular bottom is assured, which is not the case in all numerical models.

Consequently, the increment in h will be denoted by δh . It is certainly simple to later get back h^{n+1} .

By contrast it would be much more difficult to introduce the increments in u and v (the second expression for continuity would then have greater rather than lesser terms).

The particular case of gradient of a free surface with tidal flats is discussed in more detail in chapter III.8.

Discretisation in time of friction terms:

The friction terms are not linear due to $u\sqrt{u^2+v^2}$ and $v\sqrt{u^2+v^2}$. In TELEMAC-2D these terms are expressed as: $u^{n+1}\sqrt{(u^n)^2+(v^n)^2}$ and $v^{n+1}\sqrt{(u^n)^2+(v^n)^2}$. We thus obtain a totally implicit treatment of friction terms. This avoids a common defect of the explicit schemes: the inversion of velocity for large values of friction or the time step.

III.4.2 Discretisation in space :

The functions will be evaluated only at the discretisation points by decomposing the functions into bases such that:

$$f = \stackrel{n}{\bullet} f_i \psi_i$$

n is the number of discrete points, f_i is the value of the function f at point i and ψ_i is the function associated with that point.

Each of these bases is associated to a degree of freedom. This function evaluates to 1 at this point and to 0 at all the other points.

Within an element the function is interpolated from the values at the nodes of discretisation.

The interpolation functions are simple only within a reference element of which each real element is obtained from a geometrical transformation (there is one transformation for each element).

Important Note:

In TELEMAC 2D version 3.0 the spatial discretisation is the same for the variables h, u, and v. This could lead to parasitic solutions if Navier-Stokes equations were used instead of Saint-Venant equations. This is linked to the condition known as «inf-sup» or LBB (Ladyzenska•a, Brezzi, Babuska) which in principle does not exist in the case of equations for compressible fluids, which are close to Saint-Venant equations. More information can be found in [12].

Detailed study of the discretisation will now be taken up for a triangular element (In the version 3.0 of TELEMAC-2D, quadrilateral elements are dealt with by dividing them into triangles).

III.4.2.1 Linear discretisation in triangles :

This discretisation is called P1 or T3 by different authors.

The reference element is the triangle formed by the points (0,0), (1,0) and (0,1), with the numbering of the nodes as shown in the following figure:



The basis functions in this element are :

 $\begin{array}{rcl} {\rm P}_1(\xi\,,\eta) &=& (& 1\,\,-\,\,\xi\,\,-\,\,\eta\,\,\,) \\ {\rm P}_2(\xi\,,\eta) &=& (& & \xi\,\,&\,\,) \\ {\rm P}_3(\xi\,,\eta) &=& (& & & \eta\,\,\,) \end{array}$

Isoparametric transformation giving x and y as a function of ξ and η is of the form :

 $\begin{array}{l} x \ (\xi, \eta) \ = x_1 + \xi \ (x_2 \hbox{-} x_1) + \eta \ (x_3 \hbox{-} x_1) \\ y \ (\xi, \eta) \ = y_1 + \xi \ (y_2 \hbox{-} y_1) + \eta \ (y_3 \hbox{-} y_1) \end{array}$

The isoparametric transformation in triangles with linear interpolation can be easily inverted.

III.4.3 Variational Formulation:

A variational formulation with the variables u and h was established in the thesis of Nicole Goutal [7]. The depth and the velocity belong to $H_0^{-1}(\Omega)$ and $L^2(\Omega)$ respectively. $L^2(\Omega)$ is the set of functions whose square is integrable over the domain and $H_0^{-1}(\Omega)$ is a subset of $L^2(\Omega)$ of functions whose derivative is in $L^2(\Omega)$, and which are zero at the boundary of Γ . These conditions serve in establishing the existence and uniqueness of the solutions.

The proof established in [7] will not be taken up here. It is assumed that Dirichlet type boundary conditions for the velocity u are obtained; it remains to extend the results to other boundary conditions such as to the case where the Dirichlet condition applies to h and to weak impermeability.

In practice, the variational formulation consists of choosing a set ϕ_j of functions, called «test functions», and of associating each function with a discretisation point (degree of freedom j), then of substituting an equation of the form:

with:

$$\int_{\Omega} E \, \phi_j \, d\Omega = 0$$

In this chapter and foreseeing future development of the computer program, the test functions are assumed to be different from the basis functions, and that the discretisation of depth may differ from that of velocity. Therefore :

- ψ_i^h : Basis at point i for the expression of depth.
- ϕ^h_i : Test fonction at point i for the continuity equation.
- ψ_i^u : Basis at point i for the expression of the components of velocity.
- ϕ_i^u : Test function at point i of the two momentum equations.

The number of bases for the depth is denoted by nph The number of bases for the velocity is denoted by npu

npu and nph are not necessarily equal (case of a linear depth and of a quadratic velocity, for example).

As the step of advection only deletes certain terms, the general case where the advection terms appear is taken up for consideration. In order to simplify, these terms will be treated like the others which would correspond to poor centred schemes. A more detailed explanation of the

actual procedures when the method of characteristics is not implied is given in chapter III.9. The starting equations, after discretisation in time are:

Continuity :

$$\frac{h^{n+1} - h^n}{DT} + \vec{u}_{conv} \cdot \overrightarrow{grad}(h) + h_{prop} \operatorname{div}(\vec{u}) = \operatorname{Sce}$$

Momentum :

$$\frac{u^{n+1}-u^n}{DT} + \vec{u}_{conv} \cdot \vec{grad}(u) = -\theta_h g \frac{\partial(h^{n+1} - h^n)}{\partial x} - g \frac{\partial Z^n}{\partial x} + F_x + \frac{1}{h} \operatorname{div}(h \ v_e \ \vec{grad} \ u^{n+1})$$

$$\frac{v^{n+1}-v^n}{DT} + \vec{u}_{conv} \cdot \vec{grad}(v) = -\theta_h g \frac{\partial(h^{n+1} - h^n)}{\partial y} - g \frac{\partial Z^n}{\partial y} + F_y + \frac{1}{h} \operatorname{div}(h \ v_e \ \vec{grad} \ v^{n+1})$$

When h, u or v appear without a subscript n or n+1 for time, these denote semi-implicit values which have not been developed here so as to be concise. \vec{u}_{conv} and h_{prop} are explained in chapter III.4.1.

Starting from the above equations, we will proceed in 2 stages: integration over the domain with multiplication by each test function, followed by decomposing the unknown functions into bases.

1) Integration over the domain and multiplication by each test function:

Multiplication by the test functions and integration over the domain give the following equation :

For all degree of freedom for depth i between 1 and nph :

$$\int_{\Omega} (\frac{h^{n+1} - h^n}{DT}) \phi_i^h d\Omega + \int_{\Omega} \vec{u}_{conv} \cdot \vec{grad}(h) \phi_i^h d\Omega + \int_{\Omega} h_{prop} div(\vec{u}) \phi_i^h d\Omega = \int_{\Omega} Sce \phi_i^h d\Omega$$

For all degree of freedom for velocity i between 1 and npu :

$$\int_{\Omega} \left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^{n}}{DT}\right) \phi_{i}^{u} d\Omega + \int_{\Omega} \vec{\mathbf{u}}_{conv} \cdot \vec{\operatorname{grad}}(u) \phi_{i}^{u} d\Omega = -\int_{\Omega} g \theta_{h} \frac{\partial (h^{n+1} - h^{n})}{\partial x} \phi_{i}^{u} d\Omega - \int_{\Omega} g \frac{\partial Z^{n}}{\partial x} \phi_{i}^{u} d\Omega + \int_{\Omega} F_{x} \phi_{i}^{u} d\Omega + \int_{\Omega} \frac{1}{h} \operatorname{div}(h \ v_{e} \ \vec{\operatorname{grad}} u^{n+1}) \phi_{i}^{u} d\Omega$$

$$\int_{\Omega} \left(\frac{v^{n+1} - v^n}{DT} \right) \phi_i^u d\Omega + \int_{\Omega} \vec{u}_{conv} \cdot \vec{grad}(v) \phi_i^u d\Omega = - \int_{\Omega} g \theta_h \frac{\partial (h^{n+1} - h^n)}{\partial y} \phi_i^u d\Omega - \int_{\Omega} g \frac{\partial Z^n}{\partial y} \phi_i^u d\Omega + \int_{\Omega} F_y \phi_i^u d\Omega + \int_{\Omega} \frac{1}{h} \operatorname{div} (h v_e \vec{grad} v^{n+1}) \phi_i^u d\Omega$$

In order to introduce the weak boundary conditions of impermeability, the term $\int_{\Omega} h_{prop} \operatorname{div}(\vec{u}) \ \phi_i^h \ d\Omega \ of \ the \ continuity \ equation \ is \ integrated \ by \ parts \ resulting \ in :$

$$\int_{\Omega} h_{\text{prop}} \operatorname{div}(\vec{u}) \ \phi_i^h \ d\Omega = \int_{\Gamma} h_{\text{prop}} \ \vec{u} \cdot \vec{n} \ \phi_i^h \ d\Gamma - \int_{\Omega} \vec{u} \cdot \operatorname{grad}(h_{\text{prop}} \ \phi_i^h) \ d\Omega$$

With the Navier-Stokes equations or in the formulation of N. Goutal, it is the gradient of the free surface (or its equivalent the pressure gradient) which is integrated, to obtain a term which includes the depth (or the pressure).

Treatment of the term:
$$\int_{\Omega} \frac{1}{h} \operatorname{div} (h \ v_e \ \overrightarrow{\operatorname{grad}} \ u^{n+1}) \ \varphi_i^u \ d\Omega$$

In release 3.0 of TELEMAC-2D, this term is simplified and treated as :

$$\int_{\Omega} div (\nu_e \overrightarrow{\text{grad}} u^{n+1}) \phi_i^u d\Omega$$

This term also must be integrated by parts as it contains second order derivatives which do not comply with the space chosen for our functions. This results in the following terms:

$$\int_{\Omega} \operatorname{div} \left(v_{e} \operatorname{\overline{grad}} u^{n+1} \right) \varphi_{i}^{u} d\Omega = \int_{\Gamma} \varphi_{i}^{u} v_{e} \operatorname{\overline{grad}} \left(u^{n+1} \right) \cdot \overrightarrow{n} d\Gamma - \int_{\Omega} v_{e} \operatorname{\overline{grad}} \left(u^{n+1} \right) \cdot \operatorname{\overline{grad}} \left(\varphi_{i}^{u} \right) d\Omega$$

These terms provide the boundary conditions for friction.

note :

A more rigorous treatment would be to express this term as: $\frac{1}{h_i} \int_{\Omega} div (h v_e \overrightarrow{\text{grad}} u^{n+1}) \phi_i^u d\Omega$

which would enable a demonstration of the conservation of momentum (refer chapter III.7.2 for a similar demonstration of tracer mass). This method has not yet been explored but would present the problem of division by zero in the case of tidal flats.

At this juncture, the equations are of the following form:

Continuity:

$$\int_{\Omega} \left(\frac{h^{n+1} - h^{n}}{DT}\right) \phi_{i}^{h} d\Omega + \int_{\Omega} \vec{u}_{conv} \cdot \vec{grad}(h) \phi_{i}^{h} d\Omega$$
$$+ \int_{\Gamma} h_{prop} \vec{u} \cdot \vec{n} \phi_{i}^{h} d\Gamma - \int_{\Omega} \vec{u} \cdot \vec{grad}(h_{prop} \phi_{i}^{h}) d\Omega = \int_{\Omega} Sce \phi_{i}^{h} d\Omega$$

Momentum:

$$\int_{\Omega} \left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^{n}}{DT} \right) \phi_{i}^{u} d\Omega + \int_{\Omega} \vec{\mathbf{u}} \cdot \vec{\operatorname{grad}}(u) \phi_{i}^{u} d\Omega = - \int_{\Omega} g \theta_{h} \frac{\partial (h^{n+1} - h^{n})}{\partial x} \phi_{i}^{u} d\Omega - \int_{\Omega} g \frac{\partial Z^{n}}{\partial x} \phi_{i}^{u} d\Omega + \int_{\Omega} F_{x} \phi_{i}^{u} d\Omega + \int_{\Gamma} \phi_{i}^{u} v_{e} \operatorname{grad}(u^{n+1}) \cdot \vec{n} d\Gamma - \int_{\Omega} v_{e} \operatorname{grad}(u^{n+1}) \cdot \operatorname{grad}(\phi_{i}^{u}) d\Omega$$

$$\int_{\Omega} \left(\frac{v^{n+1} - v^{n}}{DT} \right) \phi_{i}^{u} d\Omega + \int_{\Omega} \vec{u} \cdot \vec{grad}(v) \phi_{i}^{u} d\Omega = - \int_{\Omega} g \theta_{h} \frac{\partial (h^{n+1} - h^{n})}{\partial y} \phi_{i}^{u} d\Omega - \int_{\Omega} g \frac{\partial Z^{n}}{\partial y} \phi_{i}^{u} d\Omega + \int_{\Omega} F_{y} \phi_{i}^{u} d\Omega + \int_{\Gamma} \phi_{i}^{u} v_{e} \vec{grad} (v^{n+1}) \cdot \vec{n} d\Gamma - \int_{\Omega} v_{e} \vec{grad} (v^{n+1}) \cdot \vec{grad} (\phi_{i}^{u}) d\Omega$$

2) Decomposition of functions into the bases :

As mentioned earlier, we now have :

$$h = \stackrel{nph}{\bullet}_{i=1} h_i \psi_i^h, \qquad u = \stackrel{npu}{\bullet}_{i=1} u_i \psi_i^u, \quad v = \stackrel{npu}{\bullet}_{i=1} v_i \psi_i^u, \quad \text{etc}$$

After this decomposition, the equations become:

Continuity:

$$\begin{array}{l} \stackrel{nph}{\bullet} (\underbrace{h_{j}^{n+1} - h_{j}^{n}}_{DT}) \int_{\Omega} \psi_{j}^{h} \phi_{i}^{h} d\Omega + \underbrace{\bullet}_{j=1}^{nph} h_{j} \int_{\Omega} \vec{u}_{conv} . \overrightarrow{grad}(\psi_{j}^{h}) \phi_{i}^{h} d\Omega \\ + \int_{\Gamma} h_{prop} \vec{u} . \vec{n} \phi_{i}^{h} d\Gamma - \underbrace{\bullet}_{j=1}^{npu} \vec{u}_{j} . \int_{\Omega} \psi_{j}^{u} \overrightarrow{grad}(h_{prop} \phi_{i}^{h}) d\Omega = \underbrace{\bullet}_{j=1}^{nph} \operatorname{Sce}_{j} \int_{\Omega} \psi_{j}^{h} \phi_{i}^{h} d\Omega$$

Momentum:

$$\int_{j=1}^{npu} (\frac{u_{j}^{n+1} - u_{j}^{n}}{DT}) \int_{\Omega} \psi_{j}^{u} \phi_{i}^{u} d\Omega + \int_{j=1}^{npu} u_{j} \int_{\Omega} \vec{u}_{conv} \cdot \vec{grad}(\psi_{j}^{u}) \phi_{i}^{u} d\Omega =$$

$$\int_{j=1}^{nph} (h_{j}^{n+1} - h_{j}^{n}) \int_{\Omega} | g \theta_{h} \frac{\partial \psi_{j}^{h}}{\partial x} \phi_{i}^{u} d\Omega - \int_{\Omega} g \frac{\partial Z^{n}}{\partial x} \phi_{i}^{u} d\Omega + \int_{\Omega} F_{x} \phi_{i}^{u} d\Omega$$

$$+ \int_{\Gamma} \phi_{i}^{u} v_{e} \vec{grad} (u^{n+1}) \cdot \vec{n} d\Gamma - \int_{j=1}^{npu} u_{j}^{n+1} \int_{\Omega} v_{e} \vec{grad} (\psi_{j}^{u}) \cdot \vec{grad} (\phi_{i}^{u}) d\Omega$$

$$\int_{j=1}^{npu} (\frac{v_{j}^{n+1} - v_{j}^{n}}{DT}) \int_{\Omega} \psi_{j}^{u} \phi_{i}^{u} d\Omega + \int_{j=1}^{npu} v_{j} \int_{\Omega} \vec{u}_{conv} \cdot \vec{grad}(\psi_{j}^{u}) \phi_{i}^{u} d\Omega =$$

$$\int_{j=1}^{nph} (h_{j}^{n+1} - h_{j}^{n}) \int_{\Omega} g \theta_{h} \frac{\partial \psi_{j}^{h}}{\partial y} \phi_{i}^{u} d\Omega - \int_{\Omega} g \frac{\partial Z^{n}}{\partial y} \phi_{i}^{u} d\Omega + \int_{\Omega} F_{y} \phi_{i}^{u} d\Omega$$

$$+ \int_{\Gamma} \phi_{i}^{u} v_{e} \vec{grad} (v^{n+1}) \cdot \vec{n} d\Gamma - \int_{j=1}^{npu} v_{j}^{n+1} \int_{\Omega} v_{e} \vec{grad} (\psi_{j}^{u}) \cdot \vec{grad} (\phi_{i}^{u}) d\Omega$$

Before converting these equations into a linear system (in paragraph II.4.5), the boundary terms will be described in greater detail.

III.4.4 Natural boundary conditions:

In the preceding equations the natural boundary conditions appear as an integral along the boundary.

<u>Continuity</u> :

In the continuity equation, the integrals along the boundary are:

$$\int_{\Gamma} h_{\text{prop}} \vec{u} \cdot \vec{n} \phi_i^h d\Gamma = \theta_u \int_{\Gamma} h_{\text{prop}} \vec{u}^{n+1} \cdot \vec{n} \phi_i^h d\Gamma + (1 - \theta_u) \int_{\Gamma} h_{\text{prop}} \vec{u}^n \cdot \vec{n} \phi_i^h d\Gamma$$

These terms are expression of the flux across the boundaries. Two possible cases are :

- If the boundary is solid, these are simply cancelled. Deleting the terms of flux across the solid boundries is known as : "weak form of impermeability condition".

- If the boundary is liquid the integral along the boundary must be calculated. There is therefore difficulty with the term:

$$\theta_{u} \int_{\Gamma} h_{prop} \vec{u}^{n+1} \cdot \vec{n} \phi_{i}^{h} d\Gamma$$
 which contains the unknown \vec{u}^{n+1} .

The results from implicitation of the term where \vec{u}^{n+1} is truly considered as an unknown have not been conclusive (case of a free outflow where parasitic reflection is observed), \vec{u}^{n+1} is therefore explicited and replaced by \vec{u}^n except if the components of velocity at time t^{n+1} are known (Dirichlet point for the velocity).

Momentum :

In each momentum equation there is only one boundary term :

$$\int_{\Gamma} \phi_i^u \ v_e \ \overrightarrow{\text{grad}} \ (u^{n+1}) \ . \ \overrightarrow{n} \ d\Gamma \ \text{or} \ \int_{\Gamma} \phi_i^u \ v_e \ \overrightarrow{\text{grad}} \ (v^{n+1}) \ . \ \overrightarrow{n} \ d\Gamma$$

or by definition:

$$\overrightarrow{grad}(u^{n+1}).\vec{n} = \frac{\partial u^{n+1}}{\partial n} \quad \text{ and } \quad \overrightarrow{grad}(v^{n+1}).\vec{n} = \frac{\partial v^{n+1}}{\partial n}$$

Therefore the boundary conditions for friction are used to calculate the boundary terms in the momentum equation:

$$\frac{\partial u^{n+1}}{\partial n} = a u^{n+1} \qquad \text{and} \qquad \frac{\partial v^{n+1}}{\partial n} = a v^{n+1}$$

The value of a is given by the user or obtained from the turbulence model.

The boundary term in the momentum equation is finally written as:

$$\int_{\Gamma} \nu_e \ a \ \vec{u}^{n+1} \ \phi_i^u \ d\Gamma$$

Unlike what was done in the continuity equation, this term is treated implicitly and appears in the matrix and not on the right hand side (see following paragraph).

Remark:

The variational formulation would also undoubtedly help to treat the boundary conditions of the following type :

$$\frac{\partial u}{\partial n} = b$$

where b is the given value. This law is the theoretical form obtained with the k-epsilon model. A treatment of this form has not so far been tried in TELEMAC. The important points to be verified for such a condition would be the stability and the possibility of an inversion of the velocities along the boundary due to the explicit character of the right hand side.

III.4.5 Matrix form of the equations obtained:

The equations obtained in paragraph II.4.3 contain only linear combinations of the unknowns, they constitute a linear system which can be written as:

$$A X = B$$

In this system, Dirichlet type points are formally considered as unknowns.

X is the vector representing the unknowns formed by 3 vectors of lengths nph, npu and npu, respectively, and contains the values of δh , u and v, vectors denoted by δH , U and V :

$$\mathbf{X} = \left(\begin{array}{c} \delta \mathbf{H} \\ \mathbf{U} \\ \mathbf{V} \end{array} \right)$$

1

B is the right hand side vector composed of three vectors called CV1, CV2 and CV3 :

$$\mathbf{B} = \begin{pmatrix} \mathbf{CV1} \\ \mathbf{CV2} \\ \mathbf{CV3} \end{pmatrix}$$

A is the matrix composed of 9 square or rectangular matrices. Some of these 9 are null matrices because the two components of velocity are not coupled. This is the consequence of weak form of the conditions for impermeability of the boundary. A is written as :

$$\mathbf{A} = \begin{vmatrix} \mathbf{A}\mathbf{M}\mathbf{1} & \mathbf{B}\mathbf{M}\mathbf{1} & \mathbf{B}\mathbf{M}\mathbf{2} \\ -\mathbf{C}\mathbf{M}\mathbf{1}^{\mathrm{T}} & \mathbf{A}\mathbf{M}\mathbf{2} & \mathbf{0} \\ -\mathbf{C}\mathbf{M}\mathbf{2}^{\mathrm{T}} & \mathbf{0} & \mathbf{A}\mathbf{M}\mathbf{3} \end{vmatrix}$$

The matrices bear the same name as the FORTRAN variables in the program. The -CM1^T form has been conserved for historical reasons. With the «celerity-velocity» formulation of equations, we get BM1 = CM1 and BM2 = CM2.

Before going on to the expressions for the matrices composing A, we will define some of the other matrices which appear in the linear system of paragraph II.4.3.:

The mass matrices:

$$M^{h}(i,j) = \int_{\Omega} \psi^{h}_{j} \phi^{h}_{i} d\Omega \quad \text{and} \quad M^{u}(i,j) = \int_{\Omega} \psi^{u}_{j} \phi^{u}_{i} d\Omega$$

The advection matrices:

$$T^{h}(i,j) = \int_{\Omega} \vec{u}_{conv} \cdot \vec{grad}(\psi_{j}^{h}) \phi_{i}^{h} d\Omega \text{ and } T^{u}(i,j) = \int_{\Omega} \vec{u}_{conv} \cdot \vec{grad}(\psi_{j}^{u}) \phi_{i}^{u} d\Omega$$

The diffusion matrix:

$$D^{u}(i,j) = \int_{\Omega} v_{e} \overrightarrow{\text{grad}} (\psi_{j}^{u}) \cdot \overrightarrow{\text{grad}} (\phi_{i}^{u}) d\Omega$$

The matrices of «gradient» or «divergence» type:

Continuity equation:

In fact the 2 following matrices are not the classical gradient matrices because of the presence of the term h_{prop} in the integrals.

$$B_{x}^{uh}(i,j) = - \int_{\Omega}^{\prod} \psi_{j}^{u} \frac{\partial}{\partial x} (h_{prop} \phi_{i}^{h}) d\Omega \qquad \text{and} \qquad B_{y}^{uh}(i,j) = - \int_{\Omega}^{\prod} \psi_{j}^{u} \frac{\partial}{\partial y} (h_{prop} \phi_{i}^{h}) d\Omega$$

(Take care of the signs - !!)

Momentum equation :

$$C_x^{uh}(i,j) = - \int_{\Omega} \frac{\partial \psi_i^h}{\partial x} \phi_j^u \, d\Omega \qquad \text{and} \qquad C_y^{uh}(i,j) = - \int_{\Omega} \frac{\partial \psi_i^h}{\partial y} \phi_j^u \, d\Omega$$

Boundary matrix:

$$F^{u}(i,j) = \int_{\Gamma} v_{e} a \psi_{j}^{u} \phi_{i}^{u} d\Gamma$$

Friction matrices:

With the chosen discretisation of time, the friction terms are (example of the component u and of ChŽzy's formula) :

$$-\int_{\Omega}\frac{1}{\cos(p)}\frac{g}{hC^{2}}u^{n+1}\sqrt{(u^{n})^{2}+(v^{n})^{2}}\phi_{i}^{u}d\Gamma$$
These terms are treated in the following simplified form :

$$- u^{n+1} \frac{1}{\cos(p_i)} \frac{g}{h_i C^2} \sqrt{(u_i^n)^2 + (v_i^n)^2} \int_{\Omega} \phi_i^u d\Gamma$$

which has the advantage of leading to diagonal matrices. A rigorous treatment is not possible because of factors such as the presence of square root for example. We could have, however, kept the complete matrices but the trials with these have shown that there was no significant difference in results. The equations in u and those in v, give the same matrices which will be denoted by F^{u} .

Blocks of matrices in the matrix A :

The matrices contained in A are then :

$$AM1 = \frac{M^h}{DT} + \theta_h T^n$$

Key word «MASS-LUMPING ON H» acts on the matrix M^h . Mass-lumping is the substitution of a matrix A with coefficients A_{ij} by a diagonal matrix D whose elements D are equal to the sum of the coefficients in the corresponding line in A :

$$D_i = \bigoplus_{j=1}^i A_{ir}.$$

In TELEMAC-2D, the mass-lumping may be partial. The key word «MASS-LUMPING ON H» is a ratio. A value of 1 corresponds to a total mass-lumping. The defaut value of 0 leaves the matrix M^{h} unchanged. Mass-lumping reinforces the diagonal of AM1 thereby improving the conditioning of the final linear system. This results though in a reduced accuracy (we move towards a formulation which approaches finite differences, while the properties of the mass conservation remain unchanged). Mass-lumping is sometimes the basic procedure employed in other Finite Element packages.

$$AM2 = AM3 = \frac{M^u}{DT} + \theta_u T^u + D^u - G^u + F^u$$

As with the depth, the key word «MASS-LUMPING ON VELOCITY» governs the mass-lumping on M^u .

 $BM1 = \theta_u B_x^{uh}$ $BM2 = \theta_u B_y^{uh}$ $CM1 = g \theta_h C_x^{uh}$ $CM2 = g \theta_h C_y^{uh}$

The matrices T^h and T^u do not figure in the charactaristics method.

The right hand sides are :

CV1 :

$$CV1 = (\theta_{h}-1) T^{n} H^{n} + (1-\theta_{u}) (B_{x}^{uh} U^{n} + B_{y}^{uh} V^{n}) + M^{h} SCE + TB1$$

TB1 representing the term:

$$-\int_{\Gamma} h_{\text{prop}} \vec{u}.\vec{n} \phi_i^n d\Gamma$$

In the characteristics method, in the expression of CV1 : $(\theta_h-1) T^n H^n$ must be replaced with $\frac{M^h}{DT} (\widetilde{H}-H^n)$

<u>CV2 :</u>

$$CV2 = \frac{M^{u}}{DT} U^{n} + (\theta_{u}-1) T^{n} U^{n} + C_{x}^{uh} Z^{n} + M^{u} F_{x}$$

In the characteristics method, in the expression of CV2 :

$$\frac{\underline{M^{u}}}{DT} U^{n} + (\theta_{u} 1) T^{n} U^{n} \text{ must be replaced with } \frac{\underline{M^{u}}}{DT} \widetilde{U}$$

<u>CV3 :</u>

$$CV3 = \frac{M^{u}}{DT} V^{n} + (\theta_{u}-1) T^{n} V^{n} + C_{y}^{uh} Z^{n} + M^{u} F_{y}$$

In the characteristics method, in the expression of CV3 :

$$\frac{M^{u}}{DT}$$
 Vⁿ + (θ_{u} -1) Tⁿ Vⁿ must be replaced with $\frac{M^{u}}{DT}$ \widetilde{V}

Note : most of the matrices of the system are dependent on time.

Once the matrices and the right hand side of the system have been constructed, it remains to take the Dirichlet boundary conditions into account and solve the linear system.

III.4.6 Solution of the linear system:

The solution of the linear system is taken over by the library BIEF; the solution involves four stages:

- taking into account, Dirichlet type boundary conditions,
- preconditioning the system to be solved (key word «PRECONDITIONING»),
- resolution by an iterative solver (key word «SOLVER»),
- getting back the unknown of the initial system (preconditioning generally implies a substitution of variables).

III.4.6.1 Dirichlet conditions:

Dirichlet points (imposed values), unlike in the classical techniques using finite elements, are not eliminated from the equations. These points are conserved in the linear system (thus avoiding renumbering the unknowns) and are assigned a value : x = imposed value. In other words, for Dirichlet points, the diagonal elements of matrix A described earlier are all made equal to1 and the non diagonal elements are cancelled. This operation is not instant for the non-assembled matrices and is described in the document [8].

III.4.6.2 Preconditioning :

Various preconditioning options are available, but for the system resulting from our propagation step diagonal preconditioning is the most efficient :

Diagonal preconditioning (also called diagonal scaling):

Let us suppose we have a system of the type AX=B.

The «diagonal-point» preconditioning, as the term suggests, is actually applied before solving the system. We construct the diagonal matrix D such that:

 $D(i,i) = \frac{1}{\sqrt{A(i,i)}}$

A(i,i) should therefore be strictly positive, which will be obtained, if necessary, by changing the sign of the equations. In the worst case the absolute value of A(i,i) is taken.

 $\mathsf{D}\mathsf{A}\mathsf{D}\mathsf{D}^{-1}\mathsf{X} = \mathsf{D}\mathsf{B}$

is then solved.

This results in:

A new matrix: A' = DAD, whose diagonal elements are equal to 1. A new unknown vector : $X' = D^{-1}X$ A new right hand side: B' = DB

After solving A'X' = B' we get back X which is equal to DX'.

c-u Preconditioning:

Another preconditioning, called c-u, is systematically applied to the propagation stage. This is based on the observation that the system resulting from the old celerity-velocity option had an excellent conditioning. We therefore try to get the same conditioning for the linear system; in the celerity-velocity option the substitution of variable was effected for the continuous variables. The general idea is to replace the unknown depth with the unknown wave celerity and to tend to the symmetry of A.

In the first step the celerity c at each point (denoted by $c^n(i)$) is calculated from the results of the previous time step or from the initial conditions. In the case of tidal flats the value of c may be set a limit to avoid division by zero. The unknown $h^{n+1}(i)$, at time t^{n+1} , is then replaced by $h^{n+1}(i) \frac{g}{2c^n(i)}$, where g is the gravitational acceleration, in the entire linear system. Finally the continuity equation is multiplied by $\frac{2g}{c^{n}(i)}$.

After solving the new system, it remains to simply find $h^{n+1}(i)$ as a function of the solution to the system which at each point is $h^{n+1}(i) \frac{g}{2c^n(i)}$.

This operation is formally equivalent to a diagonal preconditioning: the matrix A of the system is replaced by D1 A D2, where the matrices D1 and D2 are diagonal functions of c^n .

III.4.6.3 Iterative Solution :

Once preconditioned, the system is solved by a conjugate-gradient method with the solvers of BIEF library. A description of the iterative methods for the solution of linear systems is given in [9].

III.5 Solving the equations of k-epsilon model:

III.5.1 Advection step:

To solve the equations of the k- ϵ model, the fractional step method is also employed. Consequently advection is treated separately and solved together with the hydrodynamic quantities if the characteristics method is called for.

III.5.2 Production, diffusion, source terms:

III.5.2.1 Discretisation in time :

In a manner similar to the propagation step, with results of advection \tilde{k} and $\tilde{\epsilon}$ known, the equations:

$$\frac{\underline{k^{n+1}} - \widetilde{k}}{DT} - \operatorname{div}\left(\frac{\nu_{t}}{\sigma_{k}} \overrightarrow{\operatorname{grad}}(k)\right) = G + P_{kv} - \varepsilon$$

$$\frac{\varepsilon^{n+1} - \widetilde{\varepsilon}}{DT} - \operatorname{div}\left(\frac{\nu_{t}}{\sigma_{\varepsilon}} \overrightarrow{\operatorname{grad}}(\varepsilon)\right) = c_{1\varepsilon} \frac{\varepsilon}{k} G + P_{\varepsilon v} - c_{2\varepsilon} \frac{\varepsilon^{2}}{k}$$

remain to be solved.

Linearisation is effected according to the principles of semi-implicitation already explained, which leads to the system :

$$\frac{k^{n+1}}{DT} - \operatorname{div}\left(\begin{array}{c} \frac{v_t}{\sigma_k} \overrightarrow{\operatorname{grad}}(k^{n+1})\right) + \frac{\epsilon^n}{k^n} k^{n+1} = \begin{array}{c} \frac{\widetilde{k}}{DT} + G^n + P_{kv}{}^n \\ \frac{\epsilon^{n+1}}{DT} - \operatorname{div}\left(\begin{array}{c} \frac{v_t}{\sigma_{\epsilon}} \overrightarrow{\operatorname{grad}}(\epsilon^{n+1})\right) + c_{2\epsilon} \frac{\epsilon^n}{k^n} \epsilon^{n+1} = \frac{\widetilde{\epsilon}}{DT} + c_{1\epsilon} \frac{\epsilon^n}{k^n} G^{n+1} + P_{\epsilon v}^n \end{array}$$

The implicitation of certain terms on the right hand side containing ϵ may seem surprising. This is however necessary for the numerical stability of the model.

III.5.2.2 Spatial discretisation:

The variational formulation, and discretisation in space are effected exactly as for the hydrodynamic system, after integrating by parts the diffusion terms for the spatial discretisation. The boundary terms obtained in the integration by parts are assumed to be 0, which is without incidence at the entrance to the domain and at the walls since k and ε are imposed there . This conforms to a variational formulation where the space of the test functions are restrained to 0 in

the parts of the border where the unknown is given (Dirichlet condition). For a liquid boundary, this amounts to supposing that the gradients of k and ε along the normal to the exit are 0.

Conserving the same notations as before for the basic matrices (mass, diffusion), the final system can be written as :

$$\begin{pmatrix} A1 & 0 \\ 0 & A2 \end{pmatrix} \begin{pmatrix} K \\ \epsilon \end{pmatrix} = \begin{pmatrix} C1 \\ C2 \end{pmatrix}$$

with :

$$\begin{split} A1 &= (\frac{1}{DT} + \frac{\epsilon^n}{k^n}) \ M + \frac{D}{\sigma_k} \\ A2 &= (\frac{1}{DT} + c_{2\epsilon} \frac{\epsilon^n}{k^n}) \ M + \frac{D}{\sigma_\epsilon} \end{split}$$

and

$$C1 = M \left(\frac{\widetilde{K}}{DT} + G^{n} + P^{n}_{kv}\right)$$
$$C2 = M \left(\frac{\widetilde{\epsilon}}{DT} + c_{1\epsilon} \frac{\epsilon_{n}}{k_{n}} G^{n} + P^{n}_{\epsilon v}\right)$$

The two equations being uncoupled, they can be solved separately using the solver of the library BIEF.

A coupling of k and epsilon will shortly be tested to improve the stability of the model.

III.6 Tracer equation:

The tracer equation at the degree of freedom i is as follows:

$$\int_{\Omega} \frac{(T^{n+1} - T^n)}{DT} \phi_i d\Omega + \int_{\Omega} \vec{u} \cdot \vec{\text{grad}}(T) \phi_i d\Omega - \frac{1}{h_i} \int_{\Omega} \text{div}(\nu_T h \ \vec{\text{grad}}(T^{n+1})) \phi_i d\Omega = \frac{1}{h_i} \int_{\Omega} (T_{\text{sce}} - T) \ \text{Sce} \ \phi_i d\Omega$$

In this equation :

The treatment of the terms in 1/h is dictated by the considerations of the conservations of quantity of tracer (refer III.7.2).

It will be noted that the terms of diffusion are totally implicit.

The different operations performed for solving the equations of the tracer are the same as described for the k- ε model. The only difference arises from the boundary term resulting from the integration of the diffusion term, which is conserved this time, but the treatment is similar to the one described for corresponding factors in the hydrodynamic system. The boundary condition at the boundary for the tracer is:

$$\frac{\partial T}{\partial n} = a T + b$$

where a and b are real values given by the user.

For a free exit, the results of the advection step are imposed on the diffusion stage. The effects of diffusion are neglected. This procedure is undoubtably preferable to artificially imposing a solid boundary condition on a liquid boundary which would have to be done if a point belonging to a liquid exit were to be treated as a degree of freedom in the diffusion stage.

The final linear system is solved by the solver of the BIEF library.

III.7 Conservation of mass :

TELEMAC-2D optionally provides a control for the conservation of mass. The principles of such a control are described hereafter, to bring out, on the one hand, the theoretical limitations and, on the other hand, the coherence required for such a control. The demonstrations of the conservation of mass forms the guidelines and at the same time a challenge for the construction of algorithms (see for example the treatment of the Mercator projection).

III.7.1 Proof of the conservation of mass

We take up the case of the conservative continuity equation without source terms:

$$\frac{\partial h}{\partial t} + \operatorname{div}(h\vec{u}) = 0$$

After discretisation in time, variational formulation, integration by parts and solving the final linear system, we have solved for each point i where the continuity equation is treated :

$$\int_{\Omega} \frac{\mathbf{h}^{n+1} - \mathbf{h}^n}{\mathbf{D}\mathbf{T}} \, \boldsymbol{\varphi}_i \, d\Omega \, + \, \int_{\Gamma} \boldsymbol{\varphi}_i \, \mathbf{h} \, \vec{\mathbf{u}}. \vec{\mathbf{n}} \, d\Gamma \, - \, \int_{\Omega} \mathbf{h} \, \vec{\mathbf{u}}. \, \overrightarrow{\operatorname{grad}}(\boldsymbol{\varphi}_i) \, d\Omega \, = \, 0$$

We suppose here that there are no source terms.

To show the global conservation of mass, the equations for all the points are summed. Considering that:

•
$$\phi_i = 1$$
, which leads to • $\overrightarrow{\text{grad}}(\phi_i) = 0$
i

we get :

$$\int_{\Omega} (h^{n+1} - h^n) \ d\Omega = -DT \int_{\Gamma} h \vec{u} \cdot \vec{n} \ d\Gamma$$

This equation expresses that the change in the mass in the domain Ω is simply due to flux across the boundary Γ . Therefore the mass is conserved.

This formula also indicates the method to calculate the flux across boundaries, by strictly following the formulation in finite elements.

The demonstration fails when:

- The continuity has not been solved for all the points. This unfortunately happens with the conditions of imposed depths; moreover in this case the outflows are calculated at precisely the points where the continuity equation has not been solved. This explains the remanence of a residual error even when rigorously conservative schemes are employed.

and also when:

- The discretised equations cannot be brought to a conservative form. This happens when we convect h with the method of characteristics, or when the discretisations of

 \vec{u} . grad and of h div(\vec{u}) in the non-conservative continuity equation are not compatible, i.e. they cannot be transformed into a divergence operator. The incompatibility is in general caused by the discretisation in time and by the linearisation which eliminates the products of the unknown functions. This is why the use of sub-iterations where the non-linear terms are updated restores the conservation of mass.

III.7.2 Conservation of tracer

With respect to what has been said about the mass of water, things now become complicated to the point that the conservation scheme for the non-conservative equations for tracer remains a problem still partly unresolved.

The equation solved for the tracer is taken to be :

$$\begin{split} &\int_{\Omega} \frac{(T^{n+1} - T^n)}{DT} \, \phi_i d\Omega + \int_{\Omega} \vec{u}. \, \overrightarrow{\text{grad}}(T) \phi_i d\Omega - \frac{1}{h_i} \int_{\Omega} \text{div}(\nu_T h \, \overrightarrow{\text{grad}}(T^{n+1})) \phi_i d\Omega \\ &= \frac{1}{h_i} \int_{\Omega} T_{\text{sce}} \, \text{Sce} \, \phi_i \, d\Omega - \frac{1}{h_i} \int_{\Omega} T[\frac{(h^{n+1} - h^n)}{DT} + \text{div}(h \, \vec{u})] \, \phi_i d\Omega \end{split}$$

S is the source of the quantity of tracer. $T = \theta_T T^{n+1} + (1-\theta_T) T^n \text{ where } \theta_T \text{ is the implicitation coefficient of the tracer.}$ h is the water depth (1-\theta_T) h^{n+1} + \theta_T h^n (voir III.7.2). h_i is the depth h at point i.

The water discharged from the source Sce, which was on the right hand side of the continuity equation, has here been replaced by the left hand side of that same equation. This form is, as we shall see, indispensable for the scheme. Unfortunately, on one hand it must be supposed that the continuity equation has been solved in this form which implies that a rigorously conservative scheme has been employed and on the other hand that the discretisation in time of h was $(1-\theta_T)$ hⁿ⁺¹ + θ_T hⁿ. If, for example, the scheme of advection number 5 is used (key word «TYPE OF ADVECTION», refer to paragraph III.9.5) for the depth, it must be supposed that θ_T is equal to 1.

In the absence of source of water, the term $-\frac{1}{h_i} \int_{\Omega} T[\frac{(h^{n+1} - h^n)}{DT} + div(h\vec{u})] \phi_i d\Omega$ must therefore

be equal to 0. It is not possible to show the conservation of tracer if, because of approximations in the continuity equation, this term is not nil.

Strict compatibility between the continuity equation and the equation for the tracer must be ensured.

This compatibility cannot be ensured if the tracer is treated at different time steps as is the case with SUBIEF and TSEF. This is due to non-linear terms: a continuity equation solved in the discretised form between t^n and t^{n+1} , and then from t^{n+1} to t^{n+2} , does not give the same result as a single solution from t^n to t^{n+2} .

Here then is the demonstration:

Each equation for the tracer at point i is multiplied by h_{i} , then all the equations are summed. By using the expression for the decomposition of h with the test functions (which makes it therefore necessary to assume that the test functions for the tracer are the basis functions of the depth), we then get :

$$\int_{\Omega} \frac{(T^{n+1} - T^n)}{DT} h d\Omega + \int_{\Omega} h \vec{u} \cdot \vec{grad}(T) d\Omega - \int_{\Omega} div(\nu_T h \vec{grad}(T^{n+1})) d\Omega$$
$$= \int_{\Omega} T_{sce} Sce d\Omega - \int_{\Omega} T[\frac{(h^{n+1} - h^n)}{DT} + div(h\vec{u})] d\Omega$$

The terms arising from the time derivative combine to give : $\frac{(h^{n+1}T^{n+1} - h^nT^n)}{DT}$, this is the origin of the condition $h = (1-\theta_T) h^{n+1} + \theta_T h^n$.

We get finally :

$$\int_{\Omega} \frac{(h^{n+1}T^{n+1} - h^nT^n)}{DT} d\Omega + \int_{\Omega} div(hT\vec{u}) d\Omega - \int_{\Omega} div(\nu_T h \ \overrightarrow{\text{grad}}(T^{n+1})) d\Omega = \int_{\Omega} T_{\text{sce}} \text{ Sce } d\Omega$$

This equation clearly states that the variation in the quantity of the tracer in the domain is due to the flux across the boundaries and to the rate of production. It shows further that the quantity of

tracer in the domain must be calculated as $\int_{\Omega} h T d\Omega$, which may be interpreted as the scalar product (MH,T) where M is the mass matrix and H and T are the vectors containing the values of the depth and the tracer. The compatible form of the flux with the correct discretisation of h is also given.

III.8 Treatment of tidal flats and dry zones:

TELEMAC-2D offers two radically different options for treating tidal flats (key word «OPTION FOR THE TREATMENT OF TIDAL FLATS»). The first consists of treating them integrally and in the entire domain, by correcting the term which are rendered obviously false because of the absence of water, as for example the gradient of the free surface. In an exposed area, this gradient becomes the gradient of the bottom and creates parasitic driving terms as shown in the figure below in 1 dimension:



The second option consists of removing from the calculations all the elements which are not entirely wet.

III.8.1 Correction of gradient of the free surface

Treatment of term : - g $\overrightarrow{grad}(Z^n)$

A specific treatment of this term is carried out in tidal areas, where 2 problems are encountered :

- Firstly we note that the semi wet elements are characterised by a discontinuity in this gradient which passes from a value close to 0 where there is water to a value close to $\overrightarrow{\text{grad}} Z_f$ where there is none. Linear interpolation is then not suited (refer to the figure above).

- Then it is seen that for the dry elements the 2 terms likely to be non zero in the equation for the conservation of momentum are this term and friction at the bottom.

We have taken care to limiting the importance of the gradient of the free surface with respect to the others. We do not seek to perfectly model the ebb flow, particularly regarding the velocity, but make sure at best that the flow is down slope. Velocity will be properly recalculated where there is water and in the case of semi-wet elements, a slope of the free surface will be taken which will be closer to that of the wet zone than to the slope of the dry zone.

All the considerations have led to a unique formulation for partially or totally dry zones which differs from the classical expression of the gradient used elsewhere but which forms a smooth extension of the latter expression when the element tends towards total wetness.

The idea is as follows:

- The elements which require a special calculation are those in which the elevation of the bottom at one of the nodes, which we shall call «high», is greater than the elevation of the free surface at another which we shall call «low».

- Only for this calculation the elevation of the free surface at the «high» node is reduced by the difference between the elevation of the bottom at this node and the elevation of the free surface at the «low» node.

III.8.2 Masking of exposed elements

In this option we associate a mask of 1 with a wet element and 0 for a dry element (even partially dry). The contributions of the elements are multiplied by the mask which, in effect amounts to removing certain elements. The mask is relayed to the BIEF library which takes this into account when assembling.

III.9 Advection schemes :

The key-word : «TYPE OF ADVECTION» of TELEMAC-2D is a table of four values, each of which applies to a variable which is advected (in the order: velocity, depth, tracer, k-epsilon). It allows various advection schemes:

- 1 : Method of characteristics.
- 2 : "Streamline Upwind Petrov-Galerkin" (S.U.P.G.) scheme with several variants.
- 3 : «Hybrid» scheme with several variants.
- 4 : FCT (Flux-Corrected Transport)
- 5 : Variants of S.U.P.G. applied to the continuity equation.

III.9.1 The method of characteristics.

This method applied to the transport terms consists of changing to lagrangian coordinates and of taking the value of a function at time t^{n+1} at the node M as equal to the value of the same function at time t^n and at point Q obtained by retracing backwards the trajectory from point M by going back in time by a period DT. Point Q is known as "foot" of the characteristic.

For each point M, the method of characteristics comprises of 2 steps :

- Finding foot of the characteristic (point Q) by determining the trajectory which passes through M at time t^{n+1} .
- Interpolation within the element.

The order in time is a problem in the determination of the trajectory. Velocity which is required for the calculation of the trajectory is :

$$\vec{u}(t) = \vec{u^n} + \frac{(t-t^n)}{DT}\vec{u^{n+1}}$$

Unfortunately convection field at time t^{n+1} is not determined until the end of the time step. Two options are offered in TELEMAC 2D :

The advecting field at time t^n is chosen (hypothesis fixed field).

The advecting field at time $(t^{n+1} + t^n)/2$ is approached repeatedly recalculating the same time step (key word: «SUB-ITERATIONS FOR NON-LINEARITY»).

This option takes up greater computer time as it multiplies the execution time for an operation by the number of sub-iterations. We will therefore now assume a fixed field. Moreover, in the method of characteristics the order of time is complex because it depends on the calculation of the characteristics.

In TELEMAC-2D the calculation of the characteristics (trajectories) are performed by a method of Runge-Kutta of order 1 and the interpolation at the foot of the characteristic conforms

to the type of finite element chosen for the «propagation-diffusion-source terms» stage. Runge-Kutta method of order 2 has been tested but does not provide any significant improvement; it is also more expensive. Interpolation of a higher order would undoubtedly provide a greater improvement than the Runge-Kutta method of a higher order.

Calculation of the trajectory :

Each time step DT is itself divided into sub-steps DDT during which the velocity is considered to be constant. If XDEP and YDEP are the coordinates of a point D of the trajectory during calculation, the next point A of coordinates XARR,YARR is determined from the following formula (the - sign is due to the fact that we are going backwards in time from t^{n+1} to t^n):

XARR = XDEP - UDEP * DDT YARR = YDEP - VDEP * DDT

Where UDEP and VDEP are the components of the velocity vector at point D. This point being not necessarily a node in the domain, UDEP and VDEP are obtained by interpolation within the element. The essential problem with finite elements is to locate the point A, that is to determine the element to which it belongs; knowing this point is indispensable for determining the velocity at this point by interpolation. The point Q, at the foot of the characteristic, must also be located to interpolate the convected variables.

During the calculation, two events may occur before reaching time tⁿ :

* A liquid boundary is encountered: in this case in theory the trajectory continues outside the domain. As the information necessary for continuing the calculation of the trajectory is lacking, the point Q is taken at the intersection of the boundary and the trajectory. Strictly the interpolation should be done at the moment of exit which is between the time t^n and t^{n+1} . This interpolation is not done at the time being in the subroutines of TELEMAC 2D for advection.

* A solid boundary is encountered: this event, though theoretically impossible, may happen following the approximations made by the method of Runge-Kutta. The intersection between the boundary and the trajectory replaces the point A which would itself lie outside the domain. The velocity field is then projected on the boundary so as to avoid another exit from the domain at the next sub-step.

The number of time sub-steps for the Runge-Kutta method is determined locally so that each element is covered in three sub-steps.

Interpolation :

As mentioned earlier, the interpolation is compatible with the type of element selected in the second stage of the calculation. If these interpolations were found too diffusive in certain cases, higher order elements would have to be considered.

More informations about the method of charateristics may be found in [4] and [5]. The programs for advection in TELEMAC 2D are derived from CARAC [5] but have been optimized and partially vectorized.

<u>Important Note</u>: this presentation is consistant with the concept of fractional step, but does not provide any indication about the treatment of other terms normally figuring in the right hand side of the equations. These could be included in the treatment by the method of characteristics. In this case the variation of a function f along its characteristic between time t^n and t^{n+1} is not zero,

but is equal to $\int_{t^n}^{t^{n+1}}$ (other terms) $_C$ dt . The notation () $_C$ signifies that other terms apply to the particle transported by the current and are therefore taken on the characteristic curve. The necessity for a semi implicitation of all the terms which do not figure in the lagrangian derivative becomes more evident. The expression of these terms in time t^{n+1} poses no problems however it is seen that the explicit terms (at time t^n) should be taken at the foot of the characteristic.

III.9.2 Streamline Upwind Petrov Galerkin

The principle of the SUPG method perfected by T.J.R. HUGHES [10] is the use of test functions bent in the direction of current. With the method called Petrov-Galerkin, these test functions are obtained from quadratic functions; the difference in the SUPG method is the use of linear but discontinuous functions. Thus the test function ϕ_i^h is replaced by : $\phi_i^h + k \frac{\vec{u_i}}{|\vec{u_i}|} \cdot \overline{\text{grad}}(\phi_i^h)$

where $\vec{u_i}$ is the velocity at node i of the domain and k a coefficient which remains to be specified. On a one dimensional grid, the pattern of the new test functions is as follows:



In theory, these test functions must be applied to all the terms of an equation to conform to the principles of finite elements. We had to modify the method on 3 counts because of the specificity of the continuity equation: 1) The «SUPG» test function cannot be applied to all the terms of the continuity equation. In fact its application to the terms of propagation $h \operatorname{div}(\vec{u})$ seems to result in aberrant terms. There is one problem which needs to be clarified.

2) The vector $\vec{u_i}$ (equal to the nodal value of the velocity vector at point i) had to be replaced ,either by a vector $\vec{u_{el}}$ constant over each element (which could be for example the mean of the velocities at each node), or by the actual vector \vec{u} at time t^n , which is itself a variable in space. This minor change does not affect the analysis of stability but is necessary to obtain an exact conservation of mass. Both the options \vec{u} and $\vec{u_{el}}$ were programmed and \vec{u} was finally retained as it was found to be more elegant and less expensive. In the formulation of SUPG, $\vec{u_i}$ will therefore be replaced henceforth by \vec{u} .

3) The choice of k: the one recommended by Hugues is a result of an analysis of the heat equation which does not apply to our case. The choice of k will now be specified.

Choice of k :

It is well known that a upwind term of advection of second order approximation can be obtained by adding an artificial diffusion $\frac{u DX}{2}$ to a centered term.

Let us now examine the effect of the basis functions of SUPG in finite elements when applied to advection terms (we take here the advection of h) :

$$\int_{\Omega} \vec{u}_{conv}.\vec{grad}(h) \phi_i^h d\Omega \text{ becomes}:$$

$$\int_{\Omega} \vec{u}_{conv} \cdot \overrightarrow{grad}(h) \quad (\phi_i^h + k \frac{\vec{u}}{|\vec{u}|} \cdot \overrightarrow{grad}(\phi_i^h)) \ d\Omega$$

it involves adding the term:

$$\int_{\Omega} k \vec{u} \cdot \vec{u} \cdot$$

of the semi-implicitation of h.

This term is equal to, in one dimension, a diffusion of coefficient k u. It is thus seen that the diffusion $\frac{u DX}{2}$ (which transforms a centered advection term into an upwind one) is obtained for $k = \frac{DX}{2}$. Henceforth k will be written as: $k = \frac{\delta DX}{2}$, where δ is the upwind coefficient.

The key word «SUPG OPTION» offers three options for δ :

Option 0	:	$\delta = 0$. We obtain a normal (and generally unstable !) centered scheme .
Option 1	:	$\delta = 1$. We obtain an upwind scheme close to the classical SUPG. method.
Option 2	:	δ = Courant number.

The additional term resulting from SUPG can be interpreted as an artificial diffusion stabilising a centered scheme. In 2 dimension, by the effect of scalar product this diffusion applies only in the direction of the current. The option 2 which was deduced from Fourier analysis, gives the minimum diffusion to ensure the greatest stability of the scheme (voir [11]). This option is the most stable when the Courant number exceeds 1.

Having made the various choices it is possible to show that the SUPG method does not alter conservativity of our equation for continuity. Indeed the sum of the modified bases is equal to 1:

$$\stackrel{\text{Iel}}{\bullet} \phi_i^h + \stackrel{\text{Iel}}{\bullet} \delta \frac{DX}{2} \frac{\vec{u}}{|\vec{u}||} \overrightarrow{\text{grad}}(\phi_i^h) = 1 + \delta \frac{DX}{2} \frac{\vec{u}}{|\vec{u}||} \overrightarrow{\text{grad}}(\stackrel{\text{Iel}}{\bullet} \phi_i^h) = 1$$

We can further state that each term of a conservative equation can be treated with a test function of different type without altering the conservativity provided that the sum of the test functions be equal to 1 (refer [12]). For the S.U.P.G. test functions, this last property is ensured by the choice of a multiplicative coefficient δ constant for an element, and by the choice of \vec{u} in place of $\vec{u_i}$.

III.9.3 Hybrid Scheme

Premilinary remarks :

If M is a mass matrix and D a diagonal matrix obtained by summing the rows of M, a conservative smoothing of a function f is obtained by calculating $f = \frac{M}{D} f$.

This equality is written for each point i as:

$$f_{i} = \frac{\int_{\Omega}^{n} \int_{\Omega} f_{j} \psi_{i} \phi_{j} d\Omega}{\int_{\Omega} \psi_{i} d\Omega} \text{ which is equal to } \int_{\Omega} f_{i} \psi_{i} d\Omega = \int_{\Omega} f \psi_{i} d\Omega$$

Summing for all points i, we obtain:

$$\int_{\Omega} f \ d\Omega = \int_{\Omega} f \ d\Omega$$

which proves the conservation of the quantity f modified in to f.

Hybrid scheme :

The idea of the hybrid scheme is to replace the result of the method of characteristics \tilde{f} by $\tilde{f} + \frac{M}{D}(f^{cons} - \tilde{f})$, f being the function to be convected, and f^{cons} being the result of the convection of f by a conservative scheme. The difference between f^{cons} and \tilde{f} is smoothened and then added to \tilde{f} .

It can be easily shown that:

$$\int_{\Omega} f^{\text{cons}} d\Omega = \int_{\Omega} \left[\tilde{f} + \frac{M}{D} (f^{\text{cons}} - \tilde{f}) \right] d\Omega$$

This equation shows that the hybrid scheme has the same properties of conservation of mass as the conservative scheme selected for hybridization.

The stability of the scheme is discussed in [11]. In principle, this scheme is unstable if the Courant number is greater than 0.32, but in practice the propagation terms of the continuity equation make it stable.

Important note:

In the current version of TELEMAC-2D the conservative scheme chosen for hybridization is the S.U.P.G method. All the variants of SUPG are offered as alternative choices. Therefore, the key word " SUPG OPTION " is used even when the hybrid scheme is selected

Note:

The idea of hybridizing two advection schemes is also found in "FCT" (Flux Corrected Transport) schemes. In the FCT methods, a low level but monotonous scheme is hybridized with a high level scheme, and one or the other is chosen locally according to the criteria for the monotony of the result. The hybrid scheme could in future evolve towards the FCT scheme (see paragraph III.9.4).

Variants of the hybrid scheme:

These variants can be selected by the keyword «HYBRID SCHEME OPTION». The method explained so far corresponds to the option 1, it conserves the integrals of the functions, but not of their product, whereas for momentum equation, the product hu is to be conserved and not u.

<u>Option 2</u>: The product hu can be conserved by the use of a matrix M' different from M for the smoothing of the following form:

$$M'(i,j) = \int_{\Omega} h \, \psi_i \, \phi_j \, d\Omega \text{ with «lumped mass» } D'(i) = \int_{\Omega} h \, \psi_i \, d\Omega$$

The problem with this scheme which conserves momentum is that the components of D' becomes zero in the dry areas. This scheme is not applicable to tidal flats.

<u>Option 3</u>: In this case the result of the conservative scheme f^{ons} is obtained with mass-lumping or, if so preferred by a «finite difference» approach where we have an identity matrix and not a matrix for the mass. In practice f^{cons} is multiplied by $\frac{M}{D}$. before it is used.

For a given variable, the combination of the following options:

TYPE OF ADVECTION :	3	(hybrid scheme)
SUPG OPTION :	1	(SUPG with an upwindng of 1)
HYBRID SCHEME OPTION :	3	(choice of mass-lumping)

gives a conservative scheme where the Fourier analysis for Courant numbers between 0 and 1 gives the same result as obtained from the method of characteristics.

III.9.4 FCT Method.

FCT means "Flux Corrected Transport". The principle of this method is given in reference [16]. The idea is to associate two (conservative) schemes one of which is of low level and the other of high level. The low level scheme is chosen for its monotony or quasi-monotony and the high level scheme for its accuracy. We locally modulate the respective weights of the two schemes according to possibilities of oscillation and break in monotony.

The release 3.0 of TELEMAC-2D offers, for the first time, a scheme based on this principle. Both schemes are hybrid. The low level scheme corresponds to option 1 of SUPG with masslumping (the scheme that is closest to the method of characteristics as just mentioned in the previous paragraph). The higher level scheme correspond to an upwinding equal to 0.25 with the SUPG method. This is the first attempt and many more combinations can be envisaged in future.

III.9.5 Conservative scheme for the continuity equation

This scheme is based on the following conservative form of the continuity equation:

$$\frac{\dot{Z}h}{\check{Z}t} + \operatorname{div}(h\vec{u}) = \frac{\dot{Z}h}{\check{Z}t} + \operatorname{div}\left[h_{\text{prop}}\left(\theta_{u}\vec{u}^{n+1} + (1-\theta_{u})\vec{u}^{n}\right)\right] = 0$$

The S.U.P.G. test functions are employed only on the advection terms, so as to resolve:

$$\int_{\Omega} \underbrace{\check{Z}h}_{\tilde{Z}t} \varphi_{i}^{h} d\Omega + \theta_{u} \int_{\Omega} \operatorname{div} (h_{\text{prop}} \vec{u}^{n+1}) \varphi_{i}^{h} d\Omega + \theta_{u} \int_{\Omega} \vec{u}^{n+1} \cdot \overrightarrow{\text{grad}}(h_{\text{prop}}) k \underbrace{\vec{u}}_{|\vec{u}|} \cdot \overrightarrow{\text{grad}}(\varphi_{i}^{h}) d\Omega$$
$$= - (1 - \theta_{u}) \int_{\Omega} \operatorname{div} (h_{\text{prop}} \vec{u}^{n}) \varphi_{i}^{h} d\Omega - (1 - \theta_{u}) \int_{\Omega} \vec{u}^{n} \cdot \overrightarrow{\text{grad}}(h_{\text{prop}}) k \underbrace{\vec{u}}_{|\vec{u}|} \cdot \overrightarrow{\text{grad}}(\varphi_{i}^{h}) d\Omega$$

This scheme is conservative without sub-iterations at the cost of stability. In all the tested cases this scheme was found to be highly unstable for a Courant number greater than $\frac{1}{\sqrt{3}} \approx 0.6$.

NOTATION:

The notation used throughout the text is given here.

Latin alphabet:

а	:	friction coefficient on the boundaries.
avent	:	coefficient of wind drag.
b	:	coefficient of linear friction.
с	:	long wave celerity.
С	:	ChŽzy coefficient.
DT	:	time step.
F _x	:	source terms of the momentum equation in u.
Fy	:	source terms of the momentum equation in v.
g	:	gravitational acceleration.
h	:	water depth.
h _{prop}	:	propagation depth.
k	:	turbulent kinetic energy.
K	:	Strickler coefficient.
k _s	:	grain size at the bottom.
m	:	Manning coefficient.
n _{fond}	:	outward normal at the bottom ("fond" is bottom in french).
$\overline{n}_{surface}$:	outwar	rd normal at the surface.
р	:	pressure.
Pa	:	atmospheric pressure
Q _x	:	Linear discharge in m^2/s , equal to hu.
Qy	:	Linear discharge in m^2/s , equal to hv.
Qsce	:	source of outflow at the bottom in m ³ /s, specified for certain points
		on the grid .
R	:	radius of the earth.
R _{ij}	:	Reynolds tensor.
S	:	salinity.
Sce	:	source terms of the continuity equation, expressed in m/s, and
		derived from Qsce.
t	:	time.
Т	:	tracer in the Saint-Venant equations.
u	:	first component of the velocity in the Saint-Venant equations.
u _{conv}	:	first component of the advecting field.
u _{sce}	:	first component of the velocity of sources.
Uvent	:	first component of the velocity of wind.
U_1, U_2, U_3	:	components of the velocity of Navier-Stokes equations.
v	:	second component of the velocity in the Saint-Venant equations
v _{conv}	:	second component of the advecting field.
v _{sce}	:	second component of the velocity of sources.
V _{vent}	:	second component of the velocity of wind.
X	:	Mercator abscissa.

Y	:	Mercator ordinate.
Ζ	:	elevation of the free surface.
Zf	:	elevation of the bottom.

Greek alphabet:

α	:	slope angle of the bottom (by element).
ε	:	turbulent dissipation.
φ	:	longitude.
φ ₀	:	longitude of the origin.
ϕ_i^h	:	test function at point i for the continuity equation.
ϕ_i^u	:	test function at point i for the two equations for momentum.
κ	:	Karman's constant equal to 0.41.
λ	:	latitude.
λ_0	:	latitude of the origin.
ψ_i^h	:	basis at point i for expressing the depth.
ψ_i^u	:	basis at point i for expressing the components of velocity.
ν	:	molecular diffusivity in m ² /s.
v_t	:	coefficient of turbulence diffusion in m ² /s.
ve	:	coefficient of diffusion in m ² /s, which includes the dispersion and
		the turbulence.
τ_{ij}	:	viscous part of the constraints tensor.
θ	:	temperature.
θ_h	:	coefficient of semi-implicitation of the depth.
$\theta_{\rm u}$:	coefficient of semi-implicitation of the velocity.
θ_{T}	:	coefficient of semi-implicitation of the tracer.
ρ	:	density of water.
ρ_{air}	:	density of air.
	:	vector representing the rotation of the earth.

KEY WORDS OF TELEMAC-2D FIGURING IN THE TEXT

ABSCISSAE OF SOURCES 31 AIR PRESSURE 31 **BOUNDARY ROUGHNESS 48 COEFFICIENT OF WIND INFLUENCE 30** CORIOLIS 29 **CORIOLIS COEFFICIENT 29 EFFECTS OF DENSITY 37** FRICTION COEFFICIENT 27:47 **HYBRID SCHEME OPTION 84 IMPLICITATION COEFFICIENT OF TRACER 73 IMPLICITATION FOR DEPTH 54 IMPLICITATION FOR VELOCITY 53** LATITUDE OF ORIGIN POINT 32; 33 LAW OF BOTTOM FRICTION 27; 47 MASS-LUMPING ON H 67 MASS-LUMPING ON VELOCITY 67 **MEAN TEMPERATURE 37 OPTION FOR THE TREATMENT OF TIDAL FLATS 77 ORDINATES OF SOURCES 31** PRECONDITIONING 69 SOLVER 69 SOURCE VELOCITY ALONG X 31 SOURCE VELOCITY ALONG Y 31 SPHERICAL COORDINATES 29; 32; 35; 36 SUB-ITERATIONS FOR NON-LINEARITY 79 SUPG OPTION 82; 84 TIME STEP 52 **TURBULENCE MODEL FOR SOLID BOUNDARIES 48** TYPE OF ADVECTION 75; 79; 84 VALUES OF THE TRACER AT THE SOURCES 42 **VELOCITY DIFFUSIVITY 14** WATER DENSITY 38 WATER DISCHARGE OF SOURCES 31 WIND 30 WIND VELOCITY ALONG X 30 WIND VELOCITY ALONG Y 30

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